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THE CONNES CHARACTER FORMULA FOR LOCALLY COMPACT SPECTRAL TRIPLES

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Astérisque
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Institut Henri Poincaré, 11, rue Pierre et Marie Curie
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Fax: (33) 01 40 46 90 96
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Fedor Sukochev
University of New South Wales, Kensington, NSW, 2052, Australia
f.sukochev@unsw.edu.au

Dmitriy Zanin
University of New South Wales, Kensington, NSW, 2052, Australia
d.zanin@unsw.edu.au

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THE CONNES CHARACTER FORMULA FOR LOCALLY COMPACT SPECTRAL TRIPLES

by Fedor SUKOCHEV & Dmitriy ZANIN

Abstract. — A fundamental tool in noncommutative geometry is Connes' character formula. This formula is used in an essential way in the applications of noncommutative geometry to index theory and to the spectral characterisation of manifolds.

A non-compact space is modeled in noncommutative geometry by a non-unital spectral triple. Our aim is to establish the Connes character formula for non-unital spectral triples. This is significantly more difficult than in the unital case and we achieve it with the use of recently developed double operator integration techniques. Previously, only partial extensions of Connes' character formula to the non-unital case were known.

In the course of the proof, we establish two more results of importance in noncommutative geometry: an asymptotic for the heat semigroup of a non-unital spectral triple, and the analyticity of the associated ζ-function.

We require certain assumptions on the underlying spectral triple, and we verify these assumptions in the case of spectral triples associated to arbitrary complete Riemannian manifolds and also in the case of Moyal planes.

***Résumé.* (Formule du caractère de Connes pour triplets spectraux localement compacts)** – Un outil fondamental en géométrie non commutative est la formule des caractères de Connes. Cette formule est utilisée de manière essentielle dans les applications de la géométrie non commutative à la théorie de l'indice et à la caractérisation spectrale des variétés.

Un espace non compact est modélisé en géométrie non commutative par un triplet spectral sans unité. Notre objectif est d'établir la formule des caractères de Connes pour les triplets spectraux sans unité. Ceci est nettement plus difficile que dans le cas unitaire et nous y parvenons grâce à l'utilisation de techniques récentes d'intégration dites à double opérateur. Auparavant, seules des extensions partielles de la formule des caractère de Connes au cas non unitaire étaient connues.

Dans la preuve, nous établissons deux autres résultats importants en géométrie non commutative : une formule asymptotique pour le semi-groupe de chaleur d'un triplet spectral sans unité, et l'analyticité de la fonction ζ associée.

Nous exigeons certaines hypothèses sur le triplet spectral sous-jacent que nous pouvons vérifier pour tout triplet spectral associé à une variétés riemannienne complète ou à un plan de Moyal.

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Dedicated to Alain Connes
for his 70th anniversary.
With gratitude and admiration.

CHAPTER 1

INTRODUCTION

1.1. Introduction

One of the fundamental tools in noncommutative geometry is the Chern character. The Connes Character Formula (also known as the Hochschild character theorem) provides an expression for the class of the Chern character in Hochschild cohomology, and it is an important tool in the computation of the Chern character. The formula has been applied to many areas of noncommutative geometry and its applications such as the local index formula [**23**], the spectral characterisation of manifolds [**22**] and recent work in mathematical physics [**16**].

In its original formulation, [**20**], the Character Formula is stated as follows: Let $(\mathscr{A}, H, D)$ be a p-summable compact spectral triple with (possibly trivial) grading Γ (as defined in Section 2.2). By the definition of a spectral triple, for all $a \in \mathscr{A}$ the commutator $[D, a]$ has an extension to a bounded operator $\partial(a)$ on H. Furthermore, if $F = \chi_{(0,\infty)}(D) - \chi_{(-\infty,0)}(D)$ then for all $a \in \mathscr{A}$ the commutator $[F, a]$ is a compact operator in the weak Schatten ideal $\mathcal{L}_{p,\infty}$. For simplicity assume that $\ker(D) = \{0\}$, and now consider the following two linear maps on the algebraic tensor power $\mathscr{A}^{\otimes(p+1)}$, defined on an elementary tensor $c = a_0 \otimes a_1 \otimes \cdots \otimes a_p \in \mathscr{A}^{\otimes(p+1)}$ by,

$$\mathrm{Ch}(c) := \frac{1}{2}\mathrm{Tr}(\Gamma F[F, a_0][F, a_1] \cdots [F, a_p])$$

and

$$\Omega(c) := \Gamma a_0 \partial a_1 \partial a_2 \cdots \partial a_p.$$

Then the Connes Character Formula states that if c is a Hochschild cycle (as defined in Section 2.2.4) then

$$\mathrm{Tr}_\omega(\Omega(c)(1 + D^2)^{-p/2}) = \mathrm{Ch}(c)$$

for every Dixmier trace Tr_ω. In other words, the multilinear maps Ch and $c \mapsto \mathrm{Tr}_\omega(\Omega(c)(1 + D^2)^{-p/2})$ define the same class in Hochschild cohomology.

There has been great interest in generalizing the tools and results of noncommutative geometry to the “non-compact” (i.e., non-unital) setting. The definition of a spectral triple associated to a non-unital algebra originates with Connes [**21**], was furthered by the work of Rennie [**47, 48**] and Gayral, Gracia-Bondía, Iochum, Schücker and Varilly [28]. Earlier, similar ideas appeared in the work of Baaj and Julg [1]. Additional contributions to this area were made by Carey, Gayral, Rennie, and the first named author [10, 11]. The conventional definition of a non-compact spectral triple is to replace the condition that $(1+D^2)^{-1/2}$ be compact with the assumption that for all $a \in \mathscr{A}$ the operator $a(1+D^2)^{-1/2}$ is compact.

This raises an important question: is the Connes Character Formula true for locally compact spectral triples?

In this paper we are able to provide an affirmative answer to this question, provided that one assumes certain regularity properties on the spectral triple. There is a substantial difference between the theories of compact and non-compact spectral triples, in particular issues pertaining to summability are more subtle. We achieve our proof of the non-unital Character Formula using recently developed techniques of operator integration.

1.2. The main results

In this paper we prove three key theorems (Theorems 1.2.2, 1.2.3 and 1.2.5) and a new result concerning universal measurability (Theorem 1.2.7).

Essential to our approach is a certain set of assumptions on a spectral triple to be outlined below. The notion of a spectral triple, and all of the corresponding notations are explained fully in Section 2.2. By definition, if $(\mathscr{A}, H, D)$ is a spectral triple then for $a \in \mathscr{A}$, the notation $\partial(a)$ denotes the bounded extension of the commutator $[D, a]$, and for an operator T on H which preserves the domain of D, $\delta(T)$ denotes the bounded extension of $[|D|, T]$ when it exists. The notation $\mathcal{L}_{r,\infty}$, $r \geq 1$, denotes the ideal of compact operators T whose singular value sequence $\{\mu(n,T)\}_{n=0}^{\infty}$ satisfies $\mu(n,T) = O(n^{-1/r})$. The norm $\|\cdot\|_1$ is the trace-class norm.

Our main assumption on $(\mathscr{A}, H, D)$ is as follows:

Hypothesis 1.2.1. — *The spectral triple $(\mathscr{A}, H, D)$ satisfies the following conditions:*

(i) *$(\mathscr{A}, H, D)$ is a smooth spectral triple.*

(ii) *There exists $p \in \mathbb{N}$ such that $(\mathscr{A}, H, D)$ is p-dimensional, i.e., for every $a \in \mathscr{A}$,*

$$a(D+i)^{-p} \in \mathcal{L}_{1,\infty},$$
$$\partial(a)(D+i)^{-p} \in \mathcal{L}_{1,\infty}.$$

(iii) *for every $a \in \mathscr{A}$ and for all $k \geq 0$, we have*

$$\left\|\delta^k(a)(D+i\lambda)^{-p-1}\right\|_1 = O(\lambda^{-1}), \quad \lambda \to \infty,$$
$$\left\|\delta^k(\partial(a))(D+i\lambda)^{-p-1}\right\|_1 = O(\lambda^{-1}), \quad \lambda \to \infty.$$

Condition 1.2.1.(i) is well-known and widely used in the literature. The notion of "smoothness" that we use here is identical to what is sometimes referred to as QC^∞ (see Definition 2.2.7).

Condition 1.2.1.(ii) is also widely used, but we caution the reader that elsewhere in the literature an alternative definition of dimension is often used: where $(\mathscr{A}, H, D)$ is said to be p-dimensional if for all $a \in \mathscr{A}$ we have $a(D+i)^{-1} \in \mathscr{L}_{p,\infty}$ and $\partial(a)(D+i)^{-1} \in \mathscr{L}_{p,\infty}$. The definition of dimension in 1.2.1.(ii) is strictly stronger, and we discuss this issue in 2.2.3.

Condition 1.2.1.(iii) is new and specific to the locally compact situation. Indeed, if $\mathscr{A}$ is unital then 1.2.1.(iii) is redundant, as it follows from 1.2.1.(ii).

In order to show that Hypothesis 1.2.1.(iii) is reasonable, we prove that it is satisfied for spectral triples associated to the following two classes of examples:

(i) Noncommutative Euclidean spaces, a.k.a. Moyal spaces. (Section 3.3)
(ii) Complete Riemannian manifolds. (Section 3.4).

In deciding on the conditions of Hypothesis 1.2.1, we have avoided the assumption that the spectral dimension of $(\mathscr{A}, H, D)$ is isolated: this is an assumption made in **[31]**, **[23]** and in some parts of **[11]**.

Our first main result is established in Section 4.5. This result provides an asymptotic estimate of the trace of the heat operator $s \mapsto e^{-s^2D^2}$, and we remark that the following theorem is new even in the compact case.

Theorem 1.2.2. — *Let $p \in \mathbb{N}$ and let $(\mathscr{A}, H, D)$ be a spectral triple satisfying Hypothesis 1.2.1. If $c \in \mathscr{A}^{\otimes(p+1)}$ is a Hochschild cycle, then*

$$\mathrm{Tr}(\Omega(c)(1+D^2)^{1-\frac{p}{2}}e^{-s^2D^2}) = \frac{p}{2}\mathrm{Ch}(c)s^{-2} + O(s^{-1}), \quad s \downarrow 0. \tag{1.1}$$

Note that we do not require that the parity of the dimension of p match the parity of the spectral triple (i.e., p can be an odd integer while $(\mathscr{A}, H, D)$ has a nontrivial grading, and similarly p can be even while $(\mathscr{A}, H, D)$ has no grading).

Our second main result proves the analytic continuation of the ζ-function associated with the operator $(1+D^2)^{-\frac{1}{2}}$. This result recovers all previous results concerning the residue of the ζ function on a Hochschild cycle.

Theorem 1.2.3. — *Let $p \in \mathbb{N}$ and let $(\mathscr{A}, H, D)$ be a spectral triple satisfying Hypothesis 1.2.1. If $c \in \mathscr{A}^{\otimes(p+1)}$ is a Hochschild cycle, then the function*

$$\zeta_{c,D}(z) := \mathrm{Tr}(\Omega(c)(1+D^2)^{-\frac{z}{2}}), \quad \Re(z) > p \tag{1.2}$$

is holomorphic, and has analytic continuation to the set $\{\Re(z) > p-1\} \setminus \{p\}$. The point $z = p$ is a simple pole of the analytic continuation of $\zeta_{c,D}$, with corresponding residue equal to $p\mathrm{Ch}(c)$.

To prove our analogue of the Character Theorem in the unital setting, we require an additional *locality* assumption on the Hochschild cycle c. The use of locality in noncommutative geometry was pioneered by Rennie in **[48]**.

Definition 1.2.4. — *A Hochschild cycle $c = \sum_{j=1}^m a_0^j \otimes \cdots \otimes a_p^j \in \mathscr{A}^{\otimes(p+1)}$ is said to be local if there exists a positive element $\phi \in \mathscr{A}$ such that $\phi a_0^j = a_0^j$ for all $1 \leq j \leq m$.*

For example, if X is a manifold and $\mathscr{A} = C_c^\infty(X)$ is the algebra of smooth compactly supported functions on X, then every Hochschild cycle is local since we may choose ϕ to be smooth and equal to 1 on the union of the supports of $\{a_0^j\}_{j=1}^m$.

Our final result is the Connes Character Formula for locally compact spectral triples. In the compact case, our result recovers all previous results of this type (e.g., [**30**, Theorem 10.32], [**2**, Theorem 6], [**12**, Theorem 10] and [**15**, Theorem 16]).

Theorem 1.2.5. — *Let $p \in \mathbb{N}$ and let $(\mathscr{A}, H, D)$ be a spectral triple satisfying Hypothesis 1.2.1. If $c \in \mathscr{A}^{\otimes(p+1)}$ is a local Hochschild cycle, then*

$$\varphi(\Omega(c)(1+D^2)^{-\frac{p}{2}}) = \mathrm{Ch}(c), \tag{1.3}$$

for every normalized trace φ on $\mathscr{L}_{1,\infty}$.

The notion of a normalized trace on $\mathscr{L}_{1,\infty}$ is recalled in Subsection 2.1.3. The purpose of the Connes Character Formula is to compute the Hochschild class of the Chern character by a "local" formula, here stated in terms of singular traces.

A consequence of Theorem 1.2.5 being stated for arbitrary normalized traces on $\mathscr{L}_{1,\infty}$ is that we can deduce precise behavior of the distribution of eigenvalues of the operator $\Omega(c)(1+D^2)^{-p/2}$:

Corollary 1.2.6. — *Let $(\mathscr{A}, H, D)$ satisfy Hypothesis 1.2.1, and let $c \in \mathscr{A}^{\otimes(p+1)}$ be a local Hochschild cycle. Then the sequence $\{\lambda(k, \Omega(c)(1+D^2)^{-p/2})\}_{k=0}^\infty$ of eigenvalues of the operator $\Omega(c)(1+D^2)^{-p/2}$ arranged in non-increasing absolute value satisfies:*

$$\sum_{k=0}^{n} \lambda(k, \Omega(c)(1+D^2)^{-p/2}) = \mathrm{Ch}(c)\log(n) + O(1), \quad n \to \infty.$$

The above corollary is an immediate consequence of Theorem 1.2.5 and Theorem 2.1.5.

The main technical innovation of this paper concerns a certain integral representation for the difference of complex powers of positive operators, which originally appeared in [**33**] and which is reproduced here as Theorem 5.2.1.

An operator $T \in \mathscr{L}_{1,\infty}$ is called universally measurable if all normalized traces on $\mathscr{L}_{1,\infty}$ take the same value on T. A new result of this paper, and a crucial component of our proof of Theorem 1.2.5, is the following:

Theorem 1.2.7. — *Let $0 \le V \in \mathscr{L}_{1,\infty}$ and let $A \in \mathscr{L}_\infty$. Define the ζ-function:*

$$\zeta_{A,V}(z) := \mathrm{Tr}(AV^{1+z}), \quad \Re(z) > 0.$$

If there exists $\varepsilon > 0$ such that $\zeta_{A,V}$ admits an analytic continuation to the set $\{z : \Re(z) > -\varepsilon\} \setminus \{0\}$ with a simple pole at 0, then for every normalized trace φ on $\mathscr{L}_{1,\infty}$ we have:

$$\varphi(AV) = \mathrm{Res}_{z=0}\zeta_{A,V}(z).$$

In particular, AV is universally measurable.

Theorem 1.2.7 is a strengthening of an earlier result [**55**, Theorem 4.13], and a complete proof is given in Section 5.5.

1.3. Context of this paper

Connes' Character Formula dates back to Connes' 1995 paper [**20**]. There the character theorem was discovered in order to "compute by a local formula the cyclic cohomology Chern character of $(\mathscr{A}, H, D)$." Connes' work initiated a lengthy and ongoing program to strengthen, generalize and better understand the Character Formula.

Closely linked to the Character Formula is the Local Index Theorem of Connes and Moscovici [**23**], and much of the work in this field was from the point of view of index theory. Among the approaches to generalizing Connes character theorem, there is [**2**] by Benamuer and Fack, and [**12**] by Carey, Philips, Rennie and the first named author.

Instead of considering traces on $\mathscr{L}_{1,\infty}$, [**12**] deals with Dixmier traces on the Lorentz space $\mathscr{M}_{1,\infty}$. Due to an error in the statement of Lemma 14 of [**12**] which invalidates the proof in the $p = 1$ case, a followup paper [**15**] was written. In [**15**], the Character Formula is proved in the compact case for arbitrary normalized traces (rather than Dixmier traces).

During the creation of the present manuscript an oversight was located in [**15**]: in that paper the case where D has a nontrivial kernel and $(\mathscr{A}, H, D)$ is even was not handled correctly. It was incorrectly assumed in [**15**, Case 3, page 20] that if $(\mathscr{A}, H, D)$ is an even spectral triple with grading Γ, then so is

$$(\mathscr{A}, H, (\chi_{[0,\infty)}(D) - \chi_{(-\infty,0)}(D))(1 + |D|^2)^{1/2}).$$

This is false if the kernel of D is nontrivial, since then it is not necessarily the case that $\chi_{[0,\infty)}(D) - \chi_{(-\infty,0)}(D)$ anticommutes with Γ. The outcome of this oversight is that the proof of the Character Theorem as given in [**15**] is incomplete. This oversight can be corrected by using the well-known "doubling trick" that was already present in [**12**, Definition 6]. The present work supersedes that of [**15**], and so rather than submit an erratum we have decided to instead supply a complete proof here, in a more general setting.

All of the work mentioned so far in this section applies exclusively in the compact case. Adapting the tools of noncommutative geometry to the locally compact case involves substantial difficulties and this task has been heavily studied by multiple authors over the past few decades: as a small sample of this body of work we mention [**47, 48, 28, 29, 10, 11**] and more recently work by Marius Junge and Li Gao concerning noncommutative planes.

In 2000, Professor Nigel Higson published [**31**]: a detailed exposition of the local index theorem, including in the final appendix a claimed proof of the Connes Character Formula in the non-unital setting. Higson's work was a major inspiration for the present paper, since it is now understood and acknowledged by Higson that the claimed proof of the Character Formula [**31**, Theorem C.3] has a gap. This paper arose from our efforts to produce a correct statement and complete proof of the Character Formula in the non-unital setting using recently developed methods of Double Operator Integration theory.

After circulating a draft of our manuscript Carey and Rennie pointed out that there was a different way to obtain a similar result on the Hochschild class using **[11]** (which is based on **[14]**). It is proved in these papers that the "resolvent cocycle" introduced there represents the cohomology class of the Chern character. From that point of view one may obtain a different representative of the Hochschild class of the Chern character using residues of zeta functions under weaker hypotheses on the Hochschild chains and substantially stronger summability conditions on the spectral triple. For Hochschild chains satisfying some additional conditions, but not requiring locality as employed here, Carey and Rennie also have a Dixmier trace formula for the Hochschild class of the Chern character evaluated on such Hochschild chains.

1.4. Structure of the paper

This paper is structured as follows:

- Chapter 2 is devoted to preliminary definitions and concepts: we introduce the relevant definitions for operator ideals, traces, spectral triples, operator valued integrals and double operator integrals.
- Chapter 3 provides important technical properties of spectral triples. In Section 3.3 we prove that Hypothesis 1.2.1 is satisfied for the canonical spectral triple associated to noncommutative Euclidean spaces $\mathbb{R}^p_\theta$, and in Section 3.4 we show that the hypothesis is satisfied for Hodge-Dirac spectral triples associated to arbitrary complete Riemannian manifolds.
- Chapter 4 contains the proof of Theorem 1.2.2.
- Chapter 5 contains the proofs of Theorems 1.2.3, 1.2.7 and 1.2.5.
- Finally, an appendix is included to collect some of the lengthier computations.

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CHAPTER 2

PRELIMINARIES

2.1. Operators, ideals and traces

2.1.1. General notation. — Fix throughout a separable, infinite dimensional complex Hilbert space H. We denote by $\mathcal{L}_\infty$ the algebra of all bounded operators on H, with operator norm denoted $\|\cdot\|_\infty$. For a compact operator T on H, let $\lambda(T) := \{\lambda(k,T)\}_{k=0}^\infty$ denote the sequence of eigenvalues of T arranged in order of non-increasing magnitude and with multiplicities. Similarly, let $\mu(T) := \{\mu(k,T)\}_{k=0}^\infty$ denote the sequence of singular values of T, also arranged in non-increasing order with multiplicities. The kth singular value may be described equivalently as either $\mu(k,T) := \lambda(k,|T|)$ or

$$\mu(k,T) = \inf\{\|T-R\|_\infty \,:\, \mathrm{rank}(R) \leq k\}.$$

The standard trace on $\mathcal{L}_\infty$ (more precisely on the trace-class ideal) is denoted Tr.

Fix an orthonormal basis $\{e_k\}_{k=0}^\infty$ on H (the particular choice of basis is inessential). We identify the algebra ℓ_∞ of all bounded sequences with the subalgebra of diagonal operators on H with respect to the chosen basis. For a given $\alpha \in \ell_\infty$, we denote the corresponding diagonal operator by $\mathrm{diag}(\alpha)$.

For $A, B \in \mathcal{L}_\infty$, we say that B is submajorized by A in the sense of Hardy-Littlewood, written as $B \prec\prec A$, if

$$\sum_{k=0}^n \mu(k,B) \leq \sum_{k=0}^n \mu(k,A), \quad n \geq 0.$$

We say that B is logarithmically submajorized by A, written as $B \prec\prec_{\log} A$ if

$$\prod_{k=0}^n \mu(k,B) \leq \prod_{k=0}^n \mu(k,A), \quad n \geq 0.$$

An important result concerning logarithmic submajorisation is the Araki-Lieb-Thirring inequality [**34**, Theorem 2], which states that for all positive bounded operators A and B and all $r \geq 1$,

$$|AB|^r \prec\prec_{\log} A^r B^r. \tag{2.1}$$