352

ASTÉRISQUE

2013

SÉMINAIRE BOURBAKI
VOLUME 2011/2012
EXPOSÉS 1043-1058

(1053) *Space time resonances*

David LANNES

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

Séminaire BOURBAKI Mars 2012
64^e année, 2011-2012, n° 1053, p. 355 à 388

SPACE TIME RESONANCES
[after Germain, Masmoudi, Shatah]

by **David LANNES**

INTRODUCTION

An important research program in nonlinear partial differential equations consists in proving the existence of global in time smooth solutions to various nonlinear dispersive equations on $\mathbb{R}^d$ (d integer, $d \geq 1$) with small initial data. This program was initiated about three decades ago and has been the motivation for the development of powerful concepts.

A general feature is that the linear dispersive terms of the equation tend to force the solution to spread and to decay. Various *dispersive estimates* have been derived to provide precise informations on this decay. The contribution of the nonlinear terms is very different. As for ordinary differential equations, they may be responsible for the development of finite time singularities. When dealing with small data, smooth nonlinearities behave roughly as their Taylor expansion at zero. The smaller the homogeneity p of the nonlinearity at the origin, the larger the nonlinear effects. A first class of global existence results can be obtained when dispersive effects dominate nonlinear effects. Since dispersive effects increase with the dimension d, this is the general situation in large dimension and/or large p; in this situation, nonlinearities do not contribute to the large time behavior of the solution (see for instance [**39**]).

In smaller dimension or for lower order nonlinearities, the situation is more complicated and depends on the precise structure of the nonlinearity, not only on its order. For the quadratic wave equation in dimension $d = 3$, Klainerman identified [**25**] the so called *null condition* on the nonlinearities that ensures, with his powerful vector fields method, global existence for small data. This method is very robust and has been used for many other equations; a spectacular illustration is for instance [**4**] for the global nonlinear stability of the Minkowski space (see also [**30**] for a simplified proof using the notion of weak null condition). We also refer for instance to [**20**] for applications to the Schrödinger equation, to **[27]** for a small review, and to **[8]** for a new approach of the vector fields method.

Another powerful technique to obtain global existence of nonlinear dispersive equations in low dimension or lower order nonlinearities is the normal form method popularized by Shatah who used it for the nonlinear Klein-Gordon equation **[34]** (see also **[36]** for a similar approach by Simon). The idea of this method is inspired by the theory of Poincaré's normal forms for dynamical systems; for a quadratic equation for instance, it consists in making a quadratic change of unknown chosen so that the new unknown solves a cubic evolution equation, for which global existence is much easier to establish. In absence of, or with few *time resonances*, this method is very efficient, and has also been used in many works. See for instance **[33, 37, 31]**, as well as **[9]** where the relevance of null conditions for the normal form method is exploited.

In a series of papers **[14, 16, 15, 12, 13]**, Germain, Masmoudi and Shatah introduced a new method to handle situations where the normal form approach cannot be used. The same idea has also been used independently by Gustafson, Nakanishi and Tsai **[17, 18]** for the Gross-Pitaevskii equation. Working on a Duhamel formulation of the equations in Fourier variables, they identify the normal form transform as an integration by parts in time in this formula. Time resonances are the natural obstruction since they create singularities when this integration by parts is performed. Germain, Masmoudi and Shatah propose to complement this approach with an integration by parts in frequency that provides extra time decay, which is helpful to prove global well posedness. The obstructions to this approach are called by the authors *space resonances*; they differ in general from time resonances, which explains why situations that were not covered by the normal form approach can be handled this way. As for the vector fields with the null condition, the structure of the nonlinearities plays an important role for the space time resonance approach; when the nonlinearities cancel some of the singularities created by time or frequency integration by parts, one may expect the normal form method or Germain, Masmoudi and Shatah's more general approach to work even in situations where time and/or space resonances are present. Based on an analogy with optics, we call here these structural conditions *time and space transparency.*

We tried in these notes to distinguish the notion of null condition from those of space and time transparencies; we also relate them to another structural condition on the nonlinearities called *compatibility*, and which is linked to the decay rate of products of solutions of homogeneous linear dispersive equations.

Throughout these notes, we use the following quadratic wave equation as a simple example to explain Klainerman's vector field method, the normal form approach, and

Germain-Masmoudi-Shatah's new method,

$$(1) \qquad \partial_t^2 u - \Delta u = Q(\partial u, \partial u), \qquad u_{|t=0} = \varepsilon u_{(0)}, \qquad \partial_t u_{|t=0} = \varepsilon u_{(1)},$$

where $Q(\cdot,\cdot)$ is a symmetric bilinear form and $\partial u = (\partial_t u, \partial_1 u, \dots, \partial_d u)^T$. When the simplicity of this example hides important phenomena, we also use a system of two coupled such equations. We also comment on the case of general first order symmetric systems because this framework is quite adapted for a comparison of the null condition, the space and time transparencies, and the compatibility condition.

Section 1 is devoted to a general exposure of Klainerman's vector field method, while Section 2 is centered on the normal form approach. These techniques are very classical and our goal is not to review recent results related to them; we just present their basic mechanisms to help understanding the rationale and the interest of the new method of Germain, Masmoudi and Shatah, which is described in Section 3. We also include in this section a description of the authors' global existence result for the water waves equations [**16**], which is probably the most important example of application of this new method. Finally we point out in Section 4 that the null, transparency and compatibilities conditions play also a role in other contexts than the issue of global existence for small data.

1. KLAINERMAN'S VECTOR FIELDS METHOD

As explained in the introduction, global existence for small initial data is the general scenario for nonlinear dispersive equations when the dimension is large and/or the nonlinearity is of high order at the origin. For the quadratic wave equation (1), global existence is always true when $d \geq 4$. We sketch the proof of this classical result in § 1.1.

1.1. Global existence for the quadratic wave equation (1) in dimension $\mathbf{d \geq 4}$

We prove in this section the following theorem using the vector fields method introduced by Klainerman [**25**].

THEOREM 1.1 ([**25**]). — *The Cauchy problem* (1) *with smooth compactly supported initial conditions has a smooth solution for all $t \geq 0$ if $d \geq 4$ and ε is small enough.*

For the linear homogeneous wave equation,

$$(2) \qquad \partial_t^2 u - \Delta u = 0, \qquad u_{|t=0} = \varepsilon u_{(0)}, \qquad \partial_t u_{|t=0} = \varepsilon u_{(1)},$$

the energy

$$\mathscr{E}(u) = \frac{1}{2}|\partial_t u|_2^2 + \frac{1}{2}|\nabla u|_2^2$$

is conserved. More generally, one gets the following classical energy inequality after multiplying $\Box u$ by $\partial_t u$ and integrating in space,

$$\mathcal{E}(u)^{1/2}(t) \leq \mathcal{E}(u)^{1/2}(0) + \int_0^t |\Box u(\tau,\cdot)|_2 d\tau. \tag{3}$$

If Z is a vector field that commutes with the operator $\Box = \partial_t^2 - \Delta$, and if u solves (2), one also has

$$(\partial_t^2 - \Delta) Z u = 0, \tag{4}$$

and $\mathcal{E}(Zu)$ is also conserved. More generally, if $Z^1, \dots, Z^n$ is a family of vector fields that commute with the wave operator $\Box$, the quantity $\mathcal{E}(Z^1 \cdots Z^n u)$ is conserved; this yields important information on the regularity and/or decay properties of the solution. For the wave equation, the vector fields that commute with $\Box$ are

$$\partial_\alpha, \qquad Z_{jk} = x_k \partial_j - x_j \partial_k, \qquad Z_j = x_j \partial_t + t\partial_j, \tag{5}$$

where $0 \leq \alpha \leq d$, $1 \leq j,k \leq d$, $(t,x) = (x_0, x_1, \dots, x_d)$ and $\partial_\alpha = \partial_{x_\alpha}$. These vector fields correspond to invariances of the equation, respectively translation and Lorentzian invariances. Another important vector field is given by

$$Z_0 = t\partial_t + \sum_{j=1}^d x_j \partial_j, \tag{6}$$

corresponding to scaling invariance; note that Z_0 does not commute with $\Box = \partial_t^2 - \Delta$ but that $[\Box, Z_0] = 2\Box$, so that the property (4) holds. We call *commuting vector fields* the vector fields (5) and (6).

One can then build Sobolev-type norms based on these vector fields and generalize the standard embedding $H^s(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ $(s > d/2)$; more precisely, for all smooth and decaying function v of (t,x), the following Klainerman-Sobolev inequality (due to Klainerman **[25]**, see also **[21, 38]** for a proof) holds,

$$\big(1 + t + |x|\big)^{d-1} \big(1 + |t - |x||\big) |v(t,x)|^2 \leq C \sum_{|I| \leq d/2+1} |Z^I v|_2^2, \tag{7}$$

where Z^I denotes any product of $|I|$ of the above commuting vector fields.

Defining, for all $s \geq 0$, the higher order energy

$$\mathcal{E}^s(v) = \sum_{|I| \leq s} \mathcal{E}(Z^I v),$$

and remarking that for all $0 \leq \alpha \leq d$,

$$Z^I \partial_\alpha = \text{linear combination of vector fields } \partial_\beta Z^J, \text{ with } |J| \leq |I|, \tag{8}$$

the inequality (7) implies that for all product Z^K of $|K|$ commuting vector fields,

$$\big(1 + t + |x|\big)^{d-1} \big(1 + |t - |x||\big) |Z^K \partial v(t,x)|^2 \leq C \mathcal{E}^{d/2+1+|K|}(v). \tag{9}$$