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CONTINUOUS MAPPINGS BETWEEN SPACES OF ARCS

BY GOULWEN FICHOU & MASAHIRO SHIOTA

ABSTRACT. — A blow-analytic homeomorphism is an arc-analytic subanalytic homeomorphism, and therefore it induces a bijective mapping between spaces of analytic arcs. We tackle the question of the continuity of this induced mapping between the spaces of arcs, giving a positive and a negative answer depending of the topology involved. We generalise the result to spaces of definable arcs in the context of o-minimal structures, obtaining notably a uniform continuity property.

RÉSUMÉ (Applications continues entre espaces d’arcs). — Un homéomorphisme analytique après éclatements est en particulier analytique par arcs et sous-analytique, il induit donc une application bijective entre les espaces d’arcs analytiques associés. On étudie la continuité de cette application induite, en fonction de la topologie considérée. On considère également la généralisation au cadre o-minimal, obtenant ainsi une propriété de continuité uniforme.

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Introduction

The blow-analytic equivalence between real analytic function germs [12] is an interesting counterpart in the real setting to the topological equivalence between complex analytic function germs. For $f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ analytic function germs, we say that f and g are blow-analytically equivalent if there exists a blow-analytic homeomorphism germ $\phi : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ such that $f = g \circ \phi$. A homeomorphism $\phi : U \rightarrow V$ between open subsets U and V of \mathbb{R}^n is called a blow-analytic homeomorphism if there exist two finite sequences of blowings-up along smooth analytic centres $\pi : M \rightarrow U$ and $\sigma : N \rightarrow V$ and an analytic diffeomorphism $\Phi : M \rightarrow N$ such that $\phi \circ \pi = \sigma \circ \Phi$.

If the definition of blow-analytic equivalence via sequences of blowings-up makes it difficult to study, it has also very nice properties (cf. [9] for a survey). In particular a blow-analytic homeomorphism $\phi : U \rightarrow V$ is arc-analytic [13], namely if $\gamma \in \mathbb{R}\{t\}^n$ is a n -uplet of convergent power series on a neighbourhood of $0 \in \mathbb{R}$ with $\gamma(0) \in U$, then $\phi \circ \gamma$ is analytic. In particular a blow-analytic homeomorphism ϕ induces a bijective mapping ϕ_* between the spaces of analytic arcs at the origin of \mathbb{R}^n . The nice behaviour of the blow-analytic equivalence with respect to arcs and more generally spaces of arcs have already produced very interesting invariants (such as the Fukui invariants [6], zeta functions [10, 5]), a complete classification in dimension two [11], or explained some relations with respect to bi-Lipschitz property [8].

The first question we address in this paper is very natural in this context: given a blow-analytic homeomorphism ϕ , is the induced mapping ϕ_* between the spaces of analytic arcs continuous? It is natural to hope that such homeomorphism induces an homeomorphism between arcs, even if the definition of a blow-analytic homeomorphism via sequences of blowings-up makes it difficult to handle directly. We offer in this paper two answers to this question.

A first answer is that the induced mapping is not continuous, even at the level of truncated arcs, when we considered $\mathbb{R}\{t\}^n$ endowed with the product topology. We show the existence of a counter-example in dimension two in Section 1.3, where the sequence of blowings-up consists simply of the blowing-up $\pi : M \rightarrow \mathbb{R}^2$ at the origin of \mathbb{R}^2 .

A second answer is that the induced mapping is continuous... if $\mathbb{R}\{t\}^n$ is endowed with the t -adic topology (cf. Theorem 1.3)! The result is actually a simple consequence of the Hölder property of subanalytic maps. However this question has a natural generalisation in the context of o-minimal structures over a real closed field where such a Hölder property is no longer available. Nevertheless, the tameness of an o-minimal structure should guaranty that the continuity of a mapping over a real closed field continues to hold when we naturally extend the field to another real closed field, and the mapping

to a mapping over the extended field. And actually, if the Hölder property suffices to obtain the continuity in the case of subanalytic mappings over real numbers, a generalisation of Łojasiewicz Inequality to locally closed definable sets (Proposition 3.5) enable to control the behaviour of arcs in the o-minimal setting in order to keep the continuity at the level of spaces of definable arcs.

We propose moreover another approach, more natural in the non necessary locally closed case (e.g., a bijection coming from a resolution of the singularities as in Example 3.13), and following a geometric approach parallel to the model-theoretic point of view developed in [3]. In particular, we study more in details in Section 3.3 the transport of properties between the initial o-minimal structure other a given real closed field and the new o-minimal structure on the real closed field of germs of definable arcs at the origin. We obtain moreover in a very simple way a uniform continuity property in Proposition 3.17.

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1. Blow-analytic homeomorphisms and continuity

If a blow-analytic homeomorphism induces a continuous mapping at the level of spaces of arcs considered with the t -adic topology (cf. Theorem 1.3), we prove the existence of a blow-analytic homeomorphism which does not induce a continuous mapping at the level of spaces of arcs when we considered it with the product topology. The counter-example is produced in Section 1.3.

1.1. Blow-analytic homeomorphisms

DEFINITION 1.1. — Let U and V be open subsets of \mathbb{R}^n . A homeomorphism $\phi : U \rightarrow V$ is called a *blow-analytic homeomorphism* if there exist two finite sequences of blowings-up along smooth analytic centres $\pi : M \rightarrow U$ and $\sigma : N \rightarrow V$ and an analytic diffeomorphism $\Phi : M \rightarrow N$ such that $\phi \circ \pi = \sigma \circ \Phi$.

Denote by $\mathbb{R}\{t\}$ the one-variable convergent power series ring and \mathfrak{m} its maximal ideal. We consider in this section $\mathbb{R}\{t\}$ equipped either with the t -adic topology or with the product topology. We regard $\mathbb{R}\{t\}^n$ as the family of analytic curve germs $c : [0; \epsilon) \rightarrow \mathbb{R}^n$, with $\epsilon > 0 \in \mathbb{R}$, at 0 and let $\mathcal{O}_0(\mathbb{R}^n)$ denote those curve germs c with $c(0) = 0$. Set

$$\mathcal{O}_U(\mathbb{R}^n) = \{c \in \mathbb{R}\{t\}^n : c(0) \in U\}$$

for an open subset U of \mathbb{R}^n . We identify $\mathcal{O}_U(\mathbb{R}^n)$ with $U \times \mathcal{O}_0(\mathbb{R}^n)$ by the correspondence

$$\mathcal{O}_U(\mathbb{R}^n) \ni c \rightarrow (c(0), c - c(0)) \in U \times \mathcal{O}_0(\mathbb{R}^n).$$