Revue d'Histoire des Mathématiques

17th century arguments for the impossibility of the indefinite and the definite circle quadrature

Jesper Lützen

Tome 20 Fascicule 1 2014

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Publiée avec le concours du Centre national de la recherche scientifique

REVUE D'HISTOIRE DES MATHÉMATIQUES

Secrétariat :
Nathalie Christiaën
Société Mathématique de France
Institut Henri Poincaré
11, rue Pierre et Marie Curie, 75231 Paris Cedex 05
Tél. : (33) 01 44 27 67 99 / Fax : (33) 01 40 46 90 96
Mél : revues@smf.ens.fr / URL : http//smf.emath.fr/

Périodicité : La *Revue* publie deux fascicules par an, de 150 pages chacun environ.

Tarifs : Prix public Europe : 80 €; prix public hors Europe : 89 €;
prix au numéro : 43 €.
Des conditions spéciales sont accordées aux membres de la SMF.

Diffusion : SMF, Maison de la SMF, Case 916 - Luminy, 13288 Marseille Cedex 9
Hindustan Book Agency, O-131, The Shopping Mall, Arjun Marg, DLF Phase 1, Gurgaon 122002, Haryana, Inde
AMS, P.O. Box 6248, Providence, Rhode Island 02940 USA

 N° ISSN : 1262-022X

Maquette couverture : Armelle Stosskopf

Revue d'histoire des mathématiques
20 (2014), p. 211–251

17TH CENTURY ARGUMENTS FOR THE IMPOSSIBILITY OF THE INDEFINITE AND THE DEFINITE CIRCLE QUADRATURE

Jesper Lützen

Abstract. — The classical problem of the quadrature (or equivalently the rectification) of the circle enjoyed a renaissance in the second half of the 17th century. The new analytic methods provided the means for the discovery of infinite expressions of π and for the first attempts to prove impossibility statements related to the quadrature of the circle. In this paper the impossibility arguments put forward by Wallis, Gregory, Leibniz and Newton are analyzed and the controversies they gave rise to are discussed. They all deal with the impossibility of finding an algebraic expression of the area of a sector of a circle in terms of its radius and cord, or of the area of the entire circle. It is argued that the controversies were partly due to a lack of precision in the formulation of the results. The impossibility results were all part of a constructive problem solving mathematical enterprise. They were intended to show that certain solutions of the quadrature problem were the best possible because simpler (analytic) solutions were impossible.

Résumé (Arguments du xvii^{e} siècle en faveur de l'impossibilité des quadratures du cercle indéfinie et définie)

Le problème classique de la quadrature (ou de la rectification) du cercle a connu un regain d'intérêt pendant la deuxième moitié du xvii^{e} siècle. Les

Texte reçu le 10 octobre 2012, révisé le 26 février 2014 et le 10 août 2014, accepté le 10 août 2014.

J. Lützen, Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen O.

Courrier électronique : lutzen@math.ku.dk

2000 Mathematics Subject Classification : 01A45, 12-03, 51M04, 51M15.

Key words and phrases : 17th century, history of mathematics, quadrature of the circle, pi, impossibility, transcendental functions, Wallis, Gregory, Leibniz, Newton.

Mots clefs. — xvii^{e} siècle, histoire des mathématiques, quadrature du cercle, pi, impossibilité, fonctions transcendentales, Wallis, Gregory, Leibniz, Newton.

nouvelles méthodes analytiques ont permis la découverte d'expressions infinies du nombre π et ont ouvert la voie vers les premières tentatives de démonstration d'assertions d'impossibilité concernant la quadrature du cercle. Dans cet article les arguments d'impossibilité de Wallis, Gregory, Leibniz et Newton sont analysés et les controverses qu'ils ont causées sont discutées. Tous les arguments concernent l'impossibilité de trouver une expression algébrique de l'aire d'un secteur d'un cercle en termes de son rayon et de sa corde ou de l'aire du cercle entier. Les controverses sont en partie dues à l'imprécision de la formulation des résultats. Les résultats d'impossibilité sont tous issus d'une entreprise mathématique constructive. Leur but était de démontrer qu'une certaine solution de la quadrature du cercle est la meilleure possible parce que des solutions plus simples (analytiques) sont impossibles.

1. INTRODUCTION

The quadrature of the circle is one of the most long lived and widely known mathematical problems. Its history is discussed in all general histories of mathematics and it has been the subject of many specialized papers and books. The problem has two aspects: A positive constructive aspect, concerning the various solutions that have been put forward for over two millennia, and a negative aspect concerning the impossibility of finding a solution using prescribed tools. The historical literature usually deals with the positive aspect in a rather continuous way, presenting and discussing the almost uninterrupted series of solutions that have been discovered since the time of Archimedes. The impossibility question on the other hand is usually dealt with in a more discontinuous fashion, often jumping from the Greeks to Lindemann's 1882 proof of the fact that the problem cannot be solved algebraically, and hence cannot be solved by ruler and compass (see e.g Berggren et al. 2004, and Hobson 1913). This difference in the treatment of the constructive aspect and the impossibility aspect of the problem reflects to some extent the importance that mathematicians of the past have attributed to these aspects and the success with which they have dealt with them. Yet, long before 1882, arguments of impossibility concerning the quadrature of the circle had been put forward by various mathematicians. The aim of the present paper is to shed light on the 17th century contributions to the impossibility question.

I shall analyze four 17th century impossibility arguments concerning the quadrature of the circle which were put forward, respectively, by John Wallis (1616–1703), James Gregory (1638–1675), Gottfried Wilhelm Leibniz (1646–1716) and Sir Isaac Newton (1642–1727).

The 17th century contributions, in particular those of Gregory and Newton, have been discussed in earlier papers such as: ([Heinrich 1901]), ([Dehn & Hellinger 1939]), ([Scriba 1983]), ([Arnol'd 1990]), ([Pourciau 2001]) and ([Pesic 2001]). I shall build on and add to these works. In particular, I shall compare the early modern arguments with each other and consider them in the context of impossibility theorems in general. I shall investigate why the impossibility questions concerning the quadrature of the circle was studied at this time at all. In this connection, I shall argue that impossibility results were primarily aimed at casting light on constructive solutions of the quadrature problem.

Considering the lack of earlier treatments of the impossibility question, I shall address questions such as: What allowed the 17th century mathematicians to contribute to the impossibility question? Which methods did they use? What did they (try to) prove to be impossible? Here it is remarkable that construction by ruler and compass did not occupy a central position. Did they distinguish different impossibility statements concerning the quadrature of the circle? Did they agree about the distinctions? Did they accept each other's proofs, and were the proofs generally accepted by their contemporaries? Did later mathematicians (including modern mathematicians and historians of mathematics) accept the proofs (or slightly rigorized versions of them) as valid? What were the strengths and weaknesses of the impossibility proofs put forward by the17th century mathematicians? An answer to the latter question will not only explain why we do not attribute the impossibility theorems to the mathematicians of this period, but also clarify what the successors could build on and what they had to repair or add.

2. HISTORICAL BACKGROUND

The early modern contributions to the impossibility of the quadrature of the circle must be studied against the background of the earlier Greek contributions to the constructive aspects of the problem and the few earlier remarks concerning the impossibility question.

The problem of constructing a square equal in area to a given circle was formulated explicitly in ancient Greece ([Knorr 1986, 25–39]). The Greeks came up with various solutions of the problem, using mechanical (transcendental) curves such as the quadratrix and the spiral, and so did their successors. Moreover, impressively good approximations were found