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COCYCLES OVER PARTIALLY HYPERBOLIC MAPS

by

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1. Partially hyperbolic diffeomorphisms

A diffeomorphism $f\colon M \to M$ on a compact manifold M is *partially hyperbolic* if there exists a continuous, nontrivial Df-invariant splitting

$$T_x M = E^s_x \oplus E^c_x \oplus E^u_x, \quad x \in M$$

of the tangent bundle such that the derivative is a contraction along E^s and an expansion along E^u, with uniform rates, and the behavior of Df along the *center bundle* E^c is in between its behaviors along E^s and E^u, again by a uniform factor. Partial hyperbolicity is a natural generalization of the notion of uniform hyperbolicity (Anosov or even Axiom A, see [**25**]), that includes many interesting additional examples, most notably: diffeomorphisms derived from Anosov through deformation by isotopy, many affine maps on homogeneous spaces, certain skew-products over hyperbolic maps, and time-1 maps of Anosov flows. Partial hyperbolicity is an open condition, so any C^1 small perturbation of these examples is partially hyperbolic as well.

The *stable* and *unstable bundles*, E^s and E^u, are uniquely integrable; that is, there exist unique f-invariant foliations $\mathcal{W}^s$ and $\mathcal{W}^u$ tangent to E^s and E^u, respectively, at all points. The leaves of these foliations are C^k if the diffeomorphism is C^k, for any $1 \le k \le \infty$, but the foliations are usually not transversely smooth. On the other hand, if f is twice differentiable then each $\mathcal{W}^s$ and $\mathcal{W}^u$ is absolutely continuous, meaning that its holonomy maps preserve the class of zero Lebesgue measure sets. These facts go back to the pioneering work of Brin, Pesin [**6**] and Hirsch, Pugh, Shub [**15**] where partial hyperbolicity and the closely related notion of normally hyperbolic foliations were introduced.

In general, the center bundle E^c need not be integrable, and similarly for the center stable bundle $E^{cs} = E^c \oplus E^s$ and the center unstable bundle $E^{cu} = E^c \oplus E^u$. We call the diffeomorphism *dynamically coherent* if E^{cs} and E^{cu} are tangent to foliations

$\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ respectively. Then intersecting the leaves of $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$, one obtains an integral foliation $\mathcal{W}^{c}$ for the center bundle as well. As it turns out, dynamical coherence does hold in many situations of interest.

Brin, Pesin [**6**] also introduced the notion of accessibility, which has played a central role in recent developments. A partially hyperbolic diffeomorphism is called *accessible* if any two points in the ambient manifold may be joined by an *su-path*, that is, a piecewise smooth path such that every smooth subpath is contained in a single leaf of $\mathcal{W}^{s}$ or a single leaf of $\mathcal{W}^{u}$. More generally, the diffeomorphism is *essentially accessible* if, given any two sets with positive volume, one can join some point of one to some point of the other by an *su*-path.

Interest in partially hyperbolic systems was greatly renewed in the mid-nineties, with two initial goals in mind. One goal was to characterize robust (or stable) transitivity, both in discrete time and continuous time. A dynamical system is *transitive* if it possesses orbits that are dense in the whole ambient space. The best known examples are all of the known constructions of Anosov diffeomorphisms (see [**25**]). Actually, since Anosov maps form an open subset of all C^1 diffeomorphisms, these are also examples of *robust* transitivity. On the other hand, early constructions by Shub [**24**] and Mañé [**17**] showed that diffeomorphisms can be robustly transitive without being Anosov. Many other examples were found by Bonatti, Díaz [**2**] and Bonatti, Viana [**5**]. A subsequent series of works started by Díaz, Pujals, Ures [**10**] for diffeomorphisms, and Morales, Pacifico, Pujals [**18**] for flows, established that in dimension three robustness implies partial hyperbolicity (where at least two of the bundles in the partially hyperbolic splitting are non-trivial). In higher dimensions one has to replace partial hyperbolicity by a related weaker condition called existence of a dominated splitting. See [**3, 5**] and also [**4**, Chapter 7] and references therein.

Another goal, initiated by Grayson, Pugh, Shub [**14**], was to recover the original attempt by Brin, Pesin [**6**] to prove that most partially hyperbolic, volume preserving diffeomorphisms are actually ergodic. To this end, Pugh, Shub [**20**] proposed the following pair of conjectures:

Conjecture 1. — Accessibility holds for an open and dense subset of C^2 partially hyperbolic diffeomorphisms, volume preserving or not.

Conjecture 2. — A partially hyperbolic C^2 volume preserving diffeomorphism with the essential accessibility property is ergodic.

Concerning Conjecture 1, it was shown by Dolgopyat, Wilkinson [**12**] that accessibility holds for a C^1-open and dense subset of all partially hyperbolic diffeomorphisms, volume preserving or not. Moreover, Didier [**11**] proved that accessibility is C^1-open for systems with 1-dimensional center bundle. More recently, Rodriguez Hertz, Rodriguez Hertz, Ures [**23**] verified the complete conjecture for conservative systems whose center bundle is one-dimensional: accessibility is C^r-dense among C^r partially hyperbolic diffeomorphisms, for any $r \geq 1$. A version of this statement for non-conservative diffeomorphisms was obtained in [**7**]. It remains open whether C^r-density still holds when $\dim E^c > 1$.

Partial versions of Conjecture 2 were obtained by Pugh, Shub [**20, 21, 22**], assuming dynamical coherence and an additional technical condition they called center bunching. Roughly speaking, their notion of center bunching means that the diffeomorphism is close to being an *isometry* along center leaves. The best result to date on Conjecture 2 is due to Burns, Wilkinson [**8**] who proved ergodicity for any accessible, partially hyperbolic volume preserving diffeomorphism (not necessarily dynamically coherent) which is not too far from being *conformal* along center leaves. Although this property is also called center bunching, it is a lot milder than the one of Pugh, Shub. In particular, it is automatic when E^c has dimension one. Thus, the previous result contains as a corollary a complete proof of Conjecture 2 when the center bundle is one-dimensional. This corollary was also observed in [**23**].

2. Cocycles

The problems considered in this volume are situated in the following context. Let $f\colon M \to M$ be a diffeomorphism. We fix a (topological, Lie...) group H with identity element e and consider the set of all (continuous, Hölder continuous, smooth...) functions $\phi\colon M \to H$. Such a function is called a *cocycle*, for reasons that are explained in the sequel. Cocycles are objects that can be composed along orbits of f, and indeed, by the cocycle generated by ϕ we often mean the sequence ϕ_n defined by

$$\phi_n(x) = \begin{cases} \phi(f^{n-1}(x)) \cdots \phi(f(x)) \cdot \phi(x) & \text{if } n > 0, \\ \phi^{-1}(f^{-n}(x)) \cdots \phi^{-1}(f^{-2}(x)) \cdot \phi^{-1}(f^{-1}(x)) & \text{if } n < 0, \\ e & \text{if } n = 0. \end{cases}$$

An equivalent definition of a cocycle, and one that generalizes to actions of groups other than $\mathbb{Z}$, is the following. A 1-cocycle is a map $\alpha\colon \mathbb{Z} \times M \to H$ satisfying the *cocycle condition:*

$$\alpha(m+n, x) = \alpha(m, f^n(x)) \cdot \alpha(n, x), \qquad \forall n, m \in \mathbb{Z},\ x \in M. \tag{1}$$

Setting $\phi(x) = \alpha(1, x)$, we obtain from the cocycle condition that $\phi_n(x) = \alpha(n, x)$, thereby establishing the equivalence of the two notions.

There are several contexts in which cocycles arise immediately in smooth dynamics and related topics, which we now discuss.

Abelian cocycles. — The cocycle ϕ is called *abelian* when the group H is abelian. A fundamental example of an abelian cocycle is the Jacobian map $\operatorname{Jac} f\colon M \to \mathbb{R}_*$ that measures the volume distortion of a diffeomorphism $f\colon M \to M$ on a Riemannian manifold M:

$$\operatorname{Jac} f(x) = \frac{d(\mathrm{vol} \circ f)}{d\,\mathrm{vol}}(x).$$

The 1-cocycle generated by $\mathrm{Jac}\, f$ is $\alpha(n,x) = \mathrm{Jac}\, f^n(x)$; in this case the cocycle condition amounts to the composition law for Radon-Nikodym derivatives. Usually this cocycle is transformed to an additive cocycle by taking a logarithm: $\log \mathrm{Jac}\, f\colon M \to \mathbb{R}$.

Abelian cocycles appear more generally as potentials in thermodynamic formalism. In this setup, one associates to each cocycle $\phi\colon M \to \mathbb{R}$ over a dynamical system $f\colon M \to M$ one or more f-invariant probability measures μ_ϕ satisfying the variational equation

$$\int_M \phi\, d\mu_\phi + h(\mu_\phi) = \sup_\nu \left(\int_M \phi\, d\nu + h(\nu) \right),$$

where the supremum on the right is taken over all f-invariant probability measures ν, and $h(\nu)$ denotes the f-entropy of the measure ν. The functional

$$P(\phi) = \sup_\nu \left(\int_M \phi\, d\nu + h(\nu) \right),$$

called the *pressure* of ϕ, has the property that if

$$\phi - \psi = \Phi \circ f - \Phi, \tag{2}$$

for some function Φ, then $P(\phi) = P(\psi)$. Hence the measure μ_ϕ depends only on the equivalence equivalence class for the equivalence relation $\phi \sim \psi$ if and only if (2) holds. As we describe below, this equation can be viewed as a coboundary equation in the appropriate cohomology theory.

Another place in which abelian cocycles appear, this time in the context of $\mathbb{R}$-actions, is in time changes in flows. Suppose that φ_t is a flow. If $\gamma\colon M \to \mathbb{R}$, then the function $\alpha\colon \mathbb{R} \times M \to \mathbb{R}$ defined by

$$\alpha(t,x) = \int_0^t \gamma(\varphi_s(x))\, ds$$

satisfies the cocycle condition:

$$\alpha(s+t,x) = \alpha(s,\varphi_t(x)) + \alpha(t,x), \tag{3}$$

which is the natural analogue of (1) for $\mathbb{R}$-actions. In general, if $\alpha\colon \mathbb{R} \times M \to \mathbb{R}$ is an arbitrary function, then the map $\psi^\alpha\colon \mathbb{R} \times M \to M$ given by

$$\psi^\alpha(t,x) = \varphi_{\alpha(t,x)}(x)$$

will define a flow on M if and only if α satisfies (3). Here too, one has a coboundary equation which corresponds to (2) for flows:

$$\alpha(t,x) - \beta(t,x) = \int_0^t \gamma(\varphi_s(x))\, ds. \tag{4}$$

One can check that if Equation (2) is satisfied for cocycles α and β and some real-valued function γ, then the flows φ^α and φ^β are time changes of one another.

Linear cocycles. — By a *linear cocycle* we will mean a cocycle with values in a matrix group. Such non-abelian cocycles also arise naturally, most notably as derivative cocycles. Suppose that $f\colon M \to M$ is a diffeomorphism of an n-manifold M. To avoid technical issues, assume that the tangent bundle TM is trivial:

$$TM = M \times \mathbb{R}^d.$$

Then the derivative Df can be represented as a map $Df\colon M \to \mathrm{GL}(d,\mathbb{R})$ which, by the Chain Rule, satisfies the (non-abelian) cocycle condition:

$$D_x f^{n+m} = D_{f^m(x)} f^n \cdot D_x f^m.$$

(We remark that the case where TM is non-trivial can be handled with a slight generalization of the notion of cocycle, using sections of an appropriate bundle.) The group $\mathrm{GL}(d,\mathbb{R})$ can be replaced by other matrix groups, such as $\mathrm{SL}(d,\mathbb{R})$, $\mathrm{Sp}(d,\mathbb{R})$, $O(d)$, $U(d)$, etc. Such group-valued cocycles arise naturally as diffeomorphism cocycles that are volume preserving, symplectic, isometric, and so on, as well as in the study of frame flows on Riemannian manifolds.

Somewhat further afield, linear cocycles play a key role in analyzing the spectrum of the one-dimensional discrete Schrödinger operators. To any abelian cocycle ϕ over an ergodic system $f\colon M \to M$ and any $p \in M$ one can associate a one-dimensional discrete Schrödinger operator $H\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ defined by

$$H(x)_n = x_n + x_{n-1} - \phi(f^n(p))\, x_n.$$

The properties of the $SL(2,\mathbb{R})$-valued cocycles defined by

$$A_E(p) = \begin{pmatrix} E-\phi & -1 \\ 1 & 0 \end{pmatrix}$$

for different choices of the parameter $E \in \mathbb{R}$ determine the spectral properties of the operator H. For example, if this cocycle is uniformly hyperbolic for some value of E, then E lies in the resolvent set of H.

3. The central problems

We briefly outline the main questions that are addressed in the two papers in this volume.

Cohomological equation. — The *cohomological (or coboundary) equation* is

$$\phi = \Phi^{-1} \cdot (\Phi \circ f). \tag{5}$$

For abelian cocycles this is usually written:

$$\phi = \Phi \circ f - \Phi. \tag{6}$$

If such a solution exists, then ϕ is called a *coboundary.* Coboundaries are in a natural sense orthogonal to f-invariant functions: they are the image of the linear operator $\phi \mapsto \phi \circ f - \phi$, whereas the f-invariant functions are the kernel. This orthogonality statement can be made precise. For example, if f preserves a probability measure μ, then in $L^2(\mu)$ the closed subspace

$$\mathcal{Z} = \{\phi \in L^2(\mu) \,|\, \phi \circ f = f\}$$

is the orthogonal complement of the L^2-closure of the space of coboundaries:

$$\mathcal{B} = \{\phi \circ f - \phi \,|\, \phi \in L^2(\mu)\}$$

This observation, which holds in some form in other function spaces as well, gives a method for proving ergodic theorems: establish the result for functions in $\mathcal{Z}$ and in $\mathcal{B}$ and then extend from the dense set $\mathcal{Z} \oplus \mathcal{B}$ using linear algebra, maximal inequalities, and so on.

An obvious obstruction to finding a continuous solution to (6) is obtained by integrating both sides against an f-invariant probability measure μ:

$$\int_M \phi \, d\mu = \int_M (\Phi \circ f - \Phi) \, d\mu = 0.$$

The natural question then arises whether this is the only obstruction; that is, if ϕ has average 0 with respect to every f-invariant probability measure μ, then does there exist a continuous solution to (6)? For transitive hyperbolic systems, the answer is "yes," as we explain below. For rigid rotations and other uniquely ergodic systems, the answer usually depends on finer arithmetic data.

For example, suppose that f is rotation on the circle by $\alpha \in \mathbb{R}/\mathbb{Z}$. A simple Fourier analysis of (6) shows that if α is Diophantine, then for any C^∞ function ϕ of average zero there exists a C^∞ solution to (6). On the other hand, if α is Liouvillean, then there exists a C^∞ function ϕ of average zero for which there is no measurable solution. For perturbations of rigid rotations, solving (6) is a key component of KAM theory, and the issue of small divisors presents obstructions to both solving the equation and establishing regularity of its solutions.

This a basic example of cohomological theory as applied to the so-called "elliptic systems." Related to these are the parabolic systems, which include flows on surfaces, polygonal billiard flows, interval exchange transformations, horocyclic flows and flows on nilmanifolds. In these systems, which are typically uniquely ergodic or possess finitely many invariant measures, solving the cohomological equation gives information about rates of convergence for ergodic averages. The relative paucity of invariant measures leads one to look at a broader class of functionals – the f-invariant distributions – as obstructions to solving the cohomological equation.

In contrast with the elliptic and parabolic systems, hyperbolic systems have a plethora of invariant measures, for example the Dirac measures supported on periodic orbits. The basic existence theory of Livšic shows that the invariant measures present a complete set of obstructions to finding a continuous solution to (6). What is more, for transitive hyperbolic systems (for which periodic orbits are dense), the periodic measures alone constitute a complete set of obstructions. Another feature of Anosov systems is that continuous solutions are always smooth.

Livšic theory for hyperbolic systems has several interesting applications. For example, applying this theory to the log Jac cocycle, it follows immediately that a transitive Anosov diffeomorphism f preserves a smooth invariant measure if and only if for every periodic point p of period n:

$$\operatorname{Jac} f^n(p) = 1.$$

Livšic theory for Anosov flows is also an ingredient in the proof of marked length spectrum rigidity for negatively curved surfaces, see [**19, 9**].

In the second paper in this volume, this Livšic theory is extended to accessible partially hyperbolic diffeomorphisms.

The role of Lyapunov exponents. — If A is a linear cocycle over $f\colon M \to M$ with values in $\mathrm{GL}(k)$, then there is a well-defined notion of the *extremal Lyapunov exponents* of A at $p \in M$:

$$\lambda_+(A,p) := \limsup_{n\to\infty} \frac{1}{n} \log \|A_n(p)\| \quad \text{and} \quad \lambda_-(A,p) := -\limsup_{n\to\infty} \frac{1}{n} \log \|A_n(p)^{-1}\|$$

Kingman's ergodic theorem implies that if f preserves a finite measure μ, then for μ-almost every p, the limits exist and depend measurably on p; moreover, each limit is constant if μ is ergodic. More generally, Oseledec's theorem implies that μ-almost every $p \in M$, the limit

$$\lambda(A,p,v) := \lim_{n\to\infty} \frac{1}{n} \log \|A_n(p)v\|$$

exists for every $v \in \mathbb{R}^k$ and assumes finitely many values, called the *Lyapunov exponents at p*. The extremal Lyapunov exponents $\lambda_+(A,p)$ and $\lambda_-(A,p)$ coincide with the largest and smallest values of $\lambda(A,p,v)$ over all $v \in \mathbb{R}^k$.

The Lyapunov exponents carry important information about a linear cocycle. In the case of the derivative cocycle Df, non-vanishing of the Lyapunov exponents on a set of positive volume implies that f has various chaotic properties. For the Schrödinger cocycle, almost everywhere vanishing of the Lyapunov exponent (equivalently, vanishing of the extremal exponents) for a positive measure set of energies $E \in \mathbb{R}$ is equivalent to the existence of absolutely continuous spectrum for the associated operator. In the first paper in this volume, a criterion is developed to establish the *non-vanishing* of the extremal Lyapunov exponents for a linear cocycle over an accessible, volume preserving, partially hyperbolic diffeomorphism. Actually, as explained below, most of the theory extends to smooth (non-linear) cocycles.

4. The general theory

To place the preceding discussion into a larger context, we briefly describe the cohomology theory in which these cocycles fit. The abelian cohomological equations that arise in dynamical systems belong to a general cohomology theory developed to study groups. To be precise, the abelian cocycles considered above are 1-cocycles in the first cohomology group of $\mathbb{Z}$ with coefficients in a $\mathbb{Z}$-module of Hölder continuous functions on M. Let us explain what we mean by this.

Let G be a group. A *G-module* is an abelian group A together with an action of G by endomorphisms of A. In the simplest cases, A is an arbitrary abelian group and G acts trivially on A. The main example considered in dynamics arises as follows. We fix a group G acting by homeomorphisms on a space X (for example, the $\mathbb{Z}$-action generated by a single homeomorphism $f\colon X \to X$). We set A to be the space $C(X, \mathbb{R})$ of continuous, $\mathbb{R}$-valued functions on X, where the abelian group structure on A is given by pointwise addition. Then there is a natural G-action on A given by precomposition: $(g \cdot \phi)(x) = \phi(g(x))$, which makes A into a G-module. Clearly the target space $\mathbb{R}$ in this construction can be replaced by any abelian topological group. If we assume higher regularity, such as smoothness, for the G-action, then $C(X, \mathbb{R})$ can be replaced by other function spaces, such as the space of Hölder functions, or smooth functions. More generally, if V is a vector bundle over X to which the action of G extends, then we can take A to be the space of (continuous, smooth...) sections of V, such as the space of smooth vector fields on X, when X is a smooth manifold.

Now given a G-module A, we construct the cohomology groups $H^n(G, A)$ as follows. For $n \geq 0$, let $C^n(G, A)$ be the set of all functions from G^n to A, which forms an abelian group. The elements of $C^n(G, A)$ are called (inhomogeneous) n-cochains. The coboundary homomorphisms $d^n\colon C^n(G, A) \to C^{n+1}(G, A)$ are defined by

$$
\begin{aligned}
(d^n \psi)(g_1, \ldots, g_{n+1}) &= g_1 \cdot \psi(g_2, \ldots, g_{n+1}) \\
&+ \sum_{i=1}^{n} (-1)^i \psi(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1}) + (-1)^{n+1} \psi(g_1, \ldots, g_n).
\end{aligned}
$$

One can check that $d^{n+1} \circ d^n = 0$; thus, we have a cochain complex and we can compute cohomology in the standard way. The group of n-cocycles is defined by $Z^n(G, A) = \ker(d^n)$, and the group of n-coboundaries is defined by $B^0(G, A) = 0$, and

$$B^n(G, A) = d^{n-1}(C^{n-1}(G, A)), \quad n \geq 1.$$

Finally, we set $H^n(G, A) = Z^n(G, A)/B^n(G, A)$.

Going back to the dynamical setting, suppose that $f\colon X \to X$ is a homeomorphism, which generates an action of the integers $\mathbb{Z}$. Then the 0-cochains are just elements of the module $C(X, \mathbb{R})$, and any $\phi\colon X \to \mathbb{R}$ generates a 1-cochain $\alpha\colon \mathbb{Z} \to C(X, \mathbb{R})$ via the formula:

$$\alpha(n) = \phi \circ f^n.$$

It is easily checked that every such cochain is a 1-cocycle and, conversely, every 1-cocycle is generated by such a function ϕ. Indeed, the cocycle condition (1) in this setting reduces to $d^1 \alpha = 0$. Moreover the abelian cohomological Equation (6) translates in this setting to:

$$\alpha = d^0 \Phi.$$

This equation asks if the given cocycle α is trivial on cohomology. Higher order cohomology groups have been studied in the dynamical context, most notably for groups of diffeomorphisms of the circle. In this context, certain elements of H^2 generalize the notion of rotation number to non-amenable groups. See [**13**].

The non-abelian cocycles also fit into a similarly defined non-abelian cohomology theory. In this case, however, the cohomology spaces no longer carry a group structure.

5. Fibered systems

The unifying concept in this volume is that of cocycles over partially hyperbolic diffeomorphisms. Let us outline our basic approach to such systems. Linear cocycles can be studied through their induced action on the projective bundle associated to the underlying vector bundle. Similarly, the classical cohomological equation is associated to an action by translations on a trivial $\mathbb{R}$-bundle over the original space.

Both these constructions are special cases of a general notion of fibered dynamical system, acting on some bundle over the original space, possibly with low fiberwise regularity. Under suitable assumptions, the invariant (stable and unstable) foliations of the base partially hyperbolic diffeomorphism lift to invariant foliations of the fibered system. Solutions of the relevant cohomological equations correspond to sections of the fiber bundle that are saturated by the lifted foliations, a property that we call *holonomy invariance*. The rich structure of these foliations allow us to obtain strong properties for these sections, when they exist.

One main conclusion of the first paper in this volume applies when the diffeomorphism satisfies the assumptions of [**8**]: partial hyperbolicity, volume preserving and center bunching. According to Theorems D and E in this paper, in that case any measurable section which is essentially (i.e., almost everywhere) saturated under the lifted stable foliation and essentially saturated under the lifted unstable foliation coincides, almost everywhere, with some section that is saturated by both lifted foliations. Moreover, if the base diffeomorphism is accessible then such a *bi-saturated* section may be chosen to be continuous.

The goal in this first paper is to detect non-zero Lyapunov exponents for fibered systems that act smoothly on the fibers (smooth cocycles), including projective actions of linear cocycles as a special case. For this, it is convenient to consider yet another fibered system, namely the push-forward action on the space of probability measures on each fiber.

General methods going back to Ledrappier [**16**] in the linear case and extended by Avila, Viana [**1**] to the present setup, give that if the Lyapunov exponents vanish almost everywhere then there exist measurable sections that are essentially saturated by either one of the lifted foliations. In view of the previous observations, it follows (Theorems B and C in this paper) that if the Lyapunov exponents vanish almost everywhere then measurable bi-saturated sections do exist, and they may be chosen to be continuous if the base dynamics is accessible.

As it turns out, bi-saturated sections are very difficult to come by, at least in the accessible case. Indeed, given any point p in the base space, consider the group of su-loops, that is, su-paths from p to itself. Each su-loop is associated to a holonomy map on the fiber over p, and a bi-invariant section gives rise to a fixed point common to all those maps. When the base diffeomorphism is accessible, the loop group is very big, yielding a large set of obstructions to the existence of such a fixed point.

In this way one gets, in particular, that generic linear cocycles over an accessible, volume preserving, partially hyperbolic diffeomorphism have some non-vanishing extremal exponent (Theorem A of this paper).

These tools developed in the first paper to handle extremal Lyapunov exponents can be applied as well to abelian cocycles. This is the starting point for the second paper in this volume. Reinterpreting in the abelian context the results of the first paper, we obtain a reformulation in the partially hyperbolic context of two of the main conclusions of the Livšic theory: existence and measurable rigidity of solutions to the coboundary equation (Theorem A parts I and III of [**26**]). The second paper then completes the remaining task of establishing regularity of solutions to the coboundary equation (Theorem A parts II and IV of [**26**]). This gives a fairly complete extension of the main conclusions of the Livšic theory from the hyperbolic to the (accessible) partially hyperbolic context.

The task is simplified conceptually by the fibered system perspective. A solution to the coboundary equation is a bi-saturated section of the associated $\mathbb{R}$-bundle; the image of this section is invariant under the lifted stable and unstable holonomy maps. Accessibility implies that these local holonomy maps act transitively on the section, meaning that the section is homogeneous under a large groupoid of transformations. A condition on the diffeomorphism called *strong bunching* implies that the holonomy maps, while not smooth, are smooth along center directions in the base manifold. Under the strong bunching hypothesis, one can then invoke ideas from the study of transformation groups to show that the section is smooth along center directions. Smoothness of the leaves of the lifted foliation gives smoothness of the section along stable and unstable directions; combined with smoothness along center directions, this gives smoothness of the invariant section. As with the conclusions in this paper, the regularity results in [**26**] apply much more generally to saturated sections of smooth cocycles (Theorem C in [**26**]).

References

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