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On the André-Quillen Cohomology of Commutative F_2-algebras

by Paul G. Goerss*

In the late 1960's and early 1970's the authors Michel André and Daniel Quillen developed a notion of homology and cohomology for commutative rings that, in many respects, behaves much like the ordinary homology and cohomology for topological spaces. For example, one can construct long exact sequences such as Quillen's transitivity sequence [21] or a product in cohomology [2]. They further noticed that this homology could say much about the commutative ring at hand. Again, for example, Quillen conjectured that the vanishing of homology groups implied that the ring was of a particularly simple type and, recently, his conjecture has been born out by Luchezar Avramov [3].

One can approach the subject of the homology and cohomology of commutative rings from two points of view. The first is from the point of view of commutative algebra. Many authors have been interested in the following situation: let Λ be a commutative, local ring with residue field $\mathbf{k}$. Then the quotient map $\Lambda \to \mathbf{k}$ allows one to define the André-Quillen homology $H_*(\Lambda, \mathbf{k})$. In this case $H_*(\Lambda, \mathbf{k})$ is of concern to local ring theorists, and squarely in the province of commutative ring theory. This is a traditional point of view. Certainly it was adopted by André and Quillen — who, if $\mathbf{k}$ was of characteristic 0, could effectively compute $H_*(\Lambda, \mathbf{k})$ in terms of $Tor_*^{\Lambda}(\mathbf{k}, \mathbf{k})$ — and more recently, by such authors as Avramov and Stephen Halperin [4].

In this work, however, we take another viewpoint — that of homotopy

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theory. The starting point is other work of Quillen [20] on non-abelian homological algebra and homotopical algebra. One of the many advances of this work of Quillen's was to isolate exactly what was required of a category $\mathcal{C}$ so that one could make all of the familiar constructions of homotopy theory in $\mathcal{C}$. If $\mathcal{C}$ satisfies the resulting list of axioms, then $\mathcal{C}$ is called a closed model category. If, in addition, $\mathcal{C}$ has a sub-category of abelian objects $\mathbf{AB}(\mathcal{C})$ and the inclusion functor $\mathbf{AB}(\mathcal{C}) \to \mathcal{C}$ has a left adjoint, then one can define the homology of objects in $\mathcal{C}$.

The model for this sort of set-up is the category of spaces; that is, the category of simplicial sets. The abelian objects are the simplicial abelian groups, and one obtains the usual homology with integer coefficients.

In this paper, we will consider the category $s\mathcal{A}$ of simplicial, supplemented, commutative $\mathbf{F}_2$-algebras. A commutative $\mathbf{F}_2$-algebra Λ is supplemented if there is an augmentation $\epsilon : \Lambda \to \mathbf{F}_2$ so that the composite

$$\mathbf{F}_2 \xrightarrow{\eta} \Lambda \xrightarrow{\epsilon} \mathbf{F}_2$$

is the identity. Here η is, of course, the unit map. An object in $s\mathcal{A}$ is then a sequence of supplemented algebras A_n, $n \geq 0$, linked by face and degeneracy maps that satisfy the simplicial identities. The category $s\mathcal{A}$ is a closed model category, with a notion of homotopy and homology. In fact, the notion of homology is exactly that of André and Quillen. We will explore homotopy and homology together and use them to illuminate each other. Indeed, the work of A.K. Bousfield [5] and the work of William Dwyer [11] imply that we know much about homotopy in $s\mathcal{A}$ and we can use their results as a foundation for our study of homology and cohomology.

Since André and Quillen define homology and cohomology using simplicial resolutions and the like, it is a natural step to studying the category $s\mathcal{A}$.

The efficacy of this approach is this: by studying simplicial objects $A \in s\mathcal{A}$ and the homology $H_*^Q A$ and cohomology $H_Q^* A$, we can not only take advice from the commutative algebra, but also from homotopy theory — and two angles are better than one. For example, homotopy theory tell us that cohomology should support a product, and because we are working in characteristic 2, something like Steenrod operations. This is indeed the case and it is exactly these operations that explain why Quillen's fundamental