Bull. Soc. Math. France
147 (2), 2019, p. 221-257

THE SADDLE-POINT METHOD IN $\mathbb{C}^N$ AND THE GENERALIZED AIRY FUNCTIONS

BY FRANCESCO PINNA & CARLO VIOLA

ABSTRACT. — We give a new version of the saddle-point method in N complex variables, for any $N \geq 2$. We apply our theorem to the asymptotic analysis of suitable multiple integrals of Airy's type.

RÉSUMÉ (*La méthode du col dans $\mathbb{C}^N$ et les fonctions d'Airy généralisées*). — Nous donnons une nouvelle version de la méthode du col en N variables complexes, pour tout $N \geq 2$. Nous appliquons notre théorème à l'analyse asymptotique de certaines intégrales multiples du type d'Airy.

1. Introduction

1.1. The saddle-point method in $\mathbb{C}$, a generalization of Laplace's method for real integrals, yields asymptotic formulae for integrals

$$I(\tau) = \int_{\gamma} e^{\tau h(z)} g(z)\,\mathrm{d}z, \tag{1}$$

Texte reçu le 10 mai 2017, modifié le 22 juillet 2018, accepté le 31 juillet 2018.

FRANCESCO PINNA, Dipartimento di Matematica e Informatica, Università di Firenze, Viale Morgagni 67/A, 50134 Firenze, Italy • *E-mail :* fpinna@math.unifi.it
CARLO VIOLA, Dipartimento di Matematica, Università di Pisa, Largo B. Pontecorvo 5, 56127 Pisa, Italy • *E-mail :* viola@dm.unipi.it

Mathematical subject classification (2010). — 41A60, 30E15, 33C10.

Key words and phrases. — N-dimensional complex integrals, asymptotic formulae, admissible saddle-point, Airy integrals.

© Société Mathématique de France
doi:10.24033/bsmf2780

where z is a complex variable, as the real parameter τ tends to $+\infty$. In (1), γ is a path contained in an open set $\Delta \subset \mathbb{C}$ and not necessarily bounded, and $g(z)$ and $h(z)$ are holomorphic functions in Δ.

The origin of the saddle-point method can be traced back to a posthumous paper of Riemann [13]. Several authors, since the end of the nineteenth century (see, e.g., [8], [3], [2], [15]), applied the saddle-point method to integrals of type (1). The basic principle of the method, in its standard version, consists in replacing γ with a new integration path λ, equivalent to γ by Cauchy's theorem so that

$$I(\tau) = \int_{\lambda} e^{\tau h(z)} g(z)\,\mathrm{d}z, \tag{2}$$

where λ contains a 'nondegenerate' (or 'simple') saddle-point z_0 of $e^{h(z)}$, i.e., at which

$$h'(z_0) = 0, \quad h''(z_0) \neq 0, \tag{3}$$

and, along λ, $|e^{h(z)}| = \exp(\operatorname{Re} h(z))$ is maximal at z_0 and at no other point on λ. Under such conditions, and assuming $g(z_0) \neq 0$ and the integral (2) to be absolutely convergent, the main term in an asymptotic expansion of $I(\tau)$, as $\tau \to +\infty$, is determined by the values $g(z_0)$, $h(z_0)$ and $h''(z_0)$.

One of the earliest applications (in [2]) of the saddle-point method concerns the asymptotic study of the Airy function

$$\operatorname{Ai}(t) := \frac{1}{2\pi i} \int_{\gamma_1 \cup \gamma_2} \exp\left(t\zeta - \tfrac{1}{3}\zeta^3\right) \mathrm{d}\zeta \quad (t \in \mathbb{R}, t \to +\infty), \tag{4}$$

where the integration path is the union $\gamma_1 \cup \gamma_2$ of two of the three half-lines defined by

$$\gamma_k = \left\{ \varrho e^{2k\pi i/3} \;\middle|\; 0 \leq \varrho < +\infty \right\} \quad (k = 0, 1, 2). \tag{5}$$

In (4), $\gamma_1 \cup \gamma_2$ is oriented from $e^{4\pi i/3}\infty$ to $e^{2\pi i/3}\infty$. The integral (4) was introduced by Airy [1] in connection with a problem in optics, and is transformed into an integral (1) by setting

$$\zeta = \tau^{1/3} z, \quad t = \tau^{2/3} \quad (\tau > 0). \tag{6}$$

This substitution yields

$$\operatorname{Ai}(\tau^{2/3}) = \frac{\tau^{1/3}}{2\pi i} \int_{\gamma_1 \cup \gamma_2} \exp\left(\tau\left(z - \frac{1}{3}z^3\right)\right) \mathrm{d}z, \tag{7}$$

and this integral is of type (1) with $g(z) = 1$ and $h(z) = z - \frac{1}{3}z^3$. The solutions of $h'(z) = 0$ are $z = \pm 1$, and the relevant saddle-point for the integral (7) to apply the saddle-point method is seen to be $z_0 = -1$.

Similarly, let

$$\mathrm{Ai}_k(t) := \frac{1}{2\pi i} \int\limits_{\gamma_0 \cup \gamma_k} \exp\left(t\zeta - \frac{1}{3}\zeta^3\right) \mathrm{d}\zeta \quad (k = 1, 2), \tag{8}$$

with γ_0, γ_1 and γ_2 defined in (5), where the path $\gamma_0 \cup \gamma_k$ is oriented from $e^{2k\pi i/3}\infty$ to $+\infty$. With the substitution (6) we get

$$\mathrm{Ai}_k(\tau^{2/3}) = \frac{\tau^{1/3}}{2\pi i} \int\limits_{\gamma_0 \cup \gamma_k} \exp\left(\tau\left(z - \frac{1}{3}z^3\right)\right) \mathrm{d}z, \tag{9}$$

and for the integrals (9) with $k = 1, 2$ the relevant saddle-point is $z_0 = 1$.

Applying to the integrals (7) and (9) the asymptotic formula (23) below with $z_0 = -1$ and $z_0 = 1$ respectively, with $g(z) = 1$ and $f(z) = \exp(z - \frac{1}{3}z^3)$, and with $\tau = t^{3/2}$ in place of n, one easily gets, for $t \to +\infty$,

$$\mathrm{Ai}(t) \sim \frac{1}{2\sqrt{\pi}} t^{-1/4} \exp\left(-\frac{2}{3}t^{3/2}\right)$$

and

$$\mathrm{Ai}_k(t) \sim -\frac{i}{2\sqrt{\pi}} t^{-1/4} \exp\left(\frac{2}{3}t^{3/2}\right) \quad (k = 1, 2).$$

We refer to [4], pp. 279–289, or to [12], pp. 40–61, for a detailed treatment of the saddle-point method in $\mathbb{C}$ and its applications to the Airy integrals.

1.2. The problem of extending the saddle-point method to integrals

$$\int\limits_{\Gamma} e^{\tau h(z_1,\dots,z_N)} g(z_1, \dots, z_N) \, \mathrm{d}z_1 \cdots \mathrm{d}z_N \tag{10}$$

over suitable manifolds Γ in $\mathbb{C}^N$ with $N \geq 2$ was studied by Fedoryuk [6]. In [7], Chapter 1, Section 4.5, Fedoryuk gives a brief account of his method. As is well known, the complex Morse lemma ([5], Prop. 3.15, p. 142, or [7], p. 125) ensures that in a neighbourhood of a nondegenerate saddle-point $\left(z_1^{(0)}, \dots, z_N^{(0)}\right)$ of $\exp h(z_1, \dots, z_N)$ (see Definition 3.2 below) there exists a local change of variables transforming $h(z_1, \dots, z_N) - h\left(z_1^{(0)}, \dots, z_N^{(0)}\right)$ into a sum of squares. Similarly to [16], Theorem 1, pp. 480–482, using Morse's lemma one gets an expansion of the integral (10) into an asymptotic power series of τ^{-1} as $\tau \to +\infty$, provided the integration manifold Γ can be transformed into a manifold Λ equivalent to Γ by Cauchy–Poincaré's theorem, thus preserving the value of (10), containing the nondegenerate saddle-point $\left(z_1^{(0)}, \dots, z_N^{(0)}\right)$ of $\exp h(z_1, \dots, z_N)$ as an interior point, and such that

$$\max_{(z_1,\dots,z_N)\in\Lambda} \operatorname{Re} h(z_1, \dots, z_N) \tag{11}$$

is attained only at $\left(z_1^{(0)},\ldots,z_N^{(0)}\right)$. Moreover, the coefficients of such an asymptotic series can be computed using Fedoryuk's method (see [16], Theorem 2, p. 483 and [7], formula (1.61), p. 125). Thus the main difficulty to get the asymptotic expansion of (10) through Fedoryuk's method is to locate the relevant nondegenerate saddle-point $\left(z_1^{(0)},\ldots,z_N^{(0)}\right)$ and prove the existence of a manifold Λ containing $\left(z_1^{(0)},\ldots,z_N^{(0)}\right)$ and satisfying the properties above.

In order to find a constructive process to transform Γ into an equivalent manifold of 'steepest descent' for $\operatorname{Re} h(z_1,\ldots,z_N)$ thus ensuring that, on such a manifold, (11) is attained only at $\left(z_1^{(0)},\ldots,z_N^{(0)}\right)$, Fedoryuk introduced techniques from algebraic topology based on homology groups, which, beside their theoretical interest, proved to be difficult to apply in concrete examples. In fact, in an example of dimension $N=2$ arising from catastrophe theory, Ursell [14] showed the non-uniqueness of steepest descent surfaces (see also the discussion in Kaminski [11]), with the result that in most cases there is no available method to transform the integration surface Γ into an equivalent surface Λ satisfying the required properties, and not even a criterion to find towards which nondegenerate saddle-point the surface Γ should be deformed.

The main example considered by Ursell [14] is an integral in $\mathbb{C}^2$ representing a natural two-dimensional generalization of the Airy integral (4)–(7). Ursell obtained results on the asymptotic behaviour of such an integral over a surface with four nearly coincident saddle-points. His final comment is: "For two complex variables little seems to be known ... More work is needed on a method of steepest descents for two complex variables, particularly on the deformation of the two-dimensional surfaces of integration".

The main purpose of the present paper is to circumvent the difficulties involved in Fedoryuk's topological deformation process by introducing a more flexible analytic method to find the relevant nondegenerate saddle-point $\left(z_1^{(0)},\ldots,z_N^{(0)}\right)$ of $f(z_1,\ldots,z_N)$ for an N-dimensional integral

$$\int_\Gamma f(z_1,\ldots,z_N)^n g(z_1,\ldots,z_N)\,\mathrm{d}z_1\cdots\mathrm{d}z_N \quad (n\in\mathbb{N},\ n\to+\infty), \tag{12}$$

for any fixed $N\geq 2$. For the treatment of (12) with $n\in\mathbb{N}$, we need not assume $f(z_1,\ldots,z_N)\neq 0$. In Theorem 4.2 we obtain an asymptotic formula for the integral (12) under assumptions which permit us to avoid the search for an equivalent integration manifold of steepest descent for $|f(z_1,\ldots,z_N)|$. In Section 5 we give a self-contained proof of Theorem 4.2. We treat (12) as an N-times iterated integral, and we apply the one-dimensional steepest descent method to each variable successively. This allows us to dispense with the global deformation process of the integration manifold. Our method, being independent of Morse's lemma, in principle could be extended, under suitable new assumptions, to the asymptotic analysis of the integral (12) in the neighbourhood of a degenerate saddle-point of $f(z_1,\dots,z_N)$.

The applications we give in Section 6 show that in several interesting cases the assigned integration manifold Γ can rather easily be transformed into an equivalent manifold Λ satisfying the assumptions of Theorem 4.2. Our Theorem 4.2 generalizes to any dimension the result proved for $N = 2$ by Hata in [9], where the author applies his method to prove nonquadraticity measures for logarithms of suitable rational numbers and concludes the introduction with the words: "To establish the $\mathbb{C}^N$-saddle method may be an interesting problem itself".

Our result is based on the notion of 'admissible' saddle-point of f, which we introduce in Definition 3.3 below. In Remark 3.4 we show that such a notion is not essentially restrictive: up to applying a suitable invertible linear transformation of the variables $z_1,\dots,z_N$, every nondegenerate saddle-point $\left(z_1^{(0)},\dots,z_N^{(0)}\right)$ is transformed into an admissible saddle-point.

If $f(z_1,\dots,z_N)\neq 0$ and

$$f(z_1,\dots,z_N)=\exp h(z_1,\dots,z_N) \tag{13}$$

with a given holomorphic function $h(z_1,\dots,z_N)$, there is no ambiguity on the value of the logarithm of f, and hence on the power

$$f(z_1,\dots,z_N)^{\tau}=\exp\left(\tau\log f(z_1,\dots,z_N)\right)$$

for $\tau\notin\mathbb{Z}$, provided one takes $\log f(z_1,\dots,z_N)=h(z_1,\dots,z_N)$, whence

$$f(z_1,\dots,z_N)^{\tau} = \exp(\tau\, h(z_1,\dots,z_N)) \tag{14}$$

as in (10). In this case our Theorem 4.2 holds with $\tau\in\mathbb{R}$, $\tau\to+\infty$, in place of the integer exponent $n\to+\infty$ in (12).

In Section 6 we apply Theorem 4.2 to prove asymptotic formulae for N-fold Airy integrals of the type considered by Ursell [14] for $N = 2$, but without restrictions concerning the mutual distance of the saddle-points. We give a full treatment of such integrals for $N = 2$. For arbitrary N, we prove the required asymptotic formula for a suitable choice of the N integration paths.

2. The saddle-point method in $\mathbb{C}$

We briefly recall some well known aspects of the classical one-dimensional saddle-point method which will be used in the following sections. The aim of the method is to prove an asymptotic formula for an integral

$$I_n=\int_{\lambda} f(z)^n g(z)\,\mathrm{d}z \quad (n\in\mathbb{N},\ n\to+\infty), \tag{15}$$