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STRONG CONVERGENCE OF THE SPECTRUM OF RANDOM PERMUTATIONS AND ALMOST-RAMANUJAN GRAPHS
[*after* **Charles Bordenave and Benoît Collins**]

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1. Introduction

Consider the two following statements:

Independent random permutations, chosen uniformly among all permutations or all matchings of n points, are strongly asymptotically free (viewed as operators on the orthogonal of the constant vector **1***).*

versus

Random n-lifts of a fixed weighted base graph are close to being Ramanujan graphs.

They seem to belong to different mathematical landscapes, random matrix theory and free probability for the first one, theory of expander graphs for the second one. They are nevertheless two instances of the same result, due to Bordenave and Collins (2019) and that we will present hereafter. In particular, we will try to explain the meaning of the statement in each context and why it represents an important improvement with respect to the previous results, starting with the motivation from graph theory and then moving to free probability. This is not the only example of a result dealing with strong asymptotic freeness that can be applied to a completely different context and we will describe in detail, in the last part of these notes, some other applications of this notion.

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2. From Ramanujan graphs to the symmetric random permutation model

The notion of *Ramanujan graph* was initially introduced by Lubotzky, Phillips, and Sarnak (1988) for d-regular graphs. The terminology *Ramanujan* comes from the fact that the construction of the regular graphs considered by Lubotzky, Phillips, and Sarnak (1988) was based on arithmetic properties of pairs of well-chosen prime numbers.

Let $G = (V, E)$ be an undirected graph, with countable vertex set V and edge set E. An edge is a subset of V with two elements (we do not allow loops nor multiple edges). The degree of a vertex $v \in V$ is defined as

$$\deg(v) := \sum_{u \in V} \mathbf{1}_{\{u,v\} \in E}.$$

If, for every $v \in V$, $\deg(v) < \infty$, the graph is said to be *locally finite* and its *adjacency operator* A is defined as follows: for any $\psi \in \ell_c(V)$, which is the subspace of $\ell^2(V)$ of vectors with finite support,

$$A\psi(v) := \sum_{u \in V; \{u,v\} \in E} \psi(u).$$

In the case when V is a finite set, A can be seen as the usual *adjacency matrix* of G. As we are dealing with undirected graphs, the adjacency operator and matrix are self-adjoint. For any integer $d \geqslant 2$, we say that G is *d-regular* if all the vertices of G have degree d. For finite d-regular graphs with n vertices, if we denote by $\lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_{n-1}$ the eigenvalues of A in non-increasing order, one can check that $\lambda_0 = d$ and for all $j \leqslant n-1$, $|\lambda_j| \leqslant d$. If G is connected, the eigenvalue λ_0 is always simple and if G is moreover bipartite, then λ_{n-1} is equal to $-d$ and is simple. They are often called the Perron–Frobenius eigenvalues and λ_0 is associated to the constant eigenvector $\mathbf{1}$. On the other hand, if we denote by

$$\lambda(G) := \max\{|\lambda_j| \text{ such that } |\lambda_j| < d\}$$

the largest eigenvalue in absolute value which is not equal to $\pm d$, we have the following result, known as the *Alon–Boppana bound*:

Theorem 2.1 (Alon, 1986). *Let $(G_{n,d})_{n \geqslant 1}$ be any sequence of connected d-regular graphs such that, for any $n \in \mathbb{N}^*$, $G_{n,d}$ has n vertices*[(1)]*. Then*

$$\liminf_{n \to \infty} \lambda(G_{n,d}) \geqslant 2\sqrt{d-1}.$$

[(1)] It requires that $n \geqslant d+1$ and nd is even.

This leads to the following definition:

Definition 2.2 (Ramanujan graph, d-regular case)**.** A d-regular, connected, finite graph G is called *Ramanujan* if and only if every eigenvalue λ of its adjacency operator is such that $\lambda \in \{-d, d\}$ or $|\lambda| \leqslant 2\sqrt{d-1}$.

Among connected d-regular graphs, Ramanujan graphs are the graphs with maximal spectral gap. For this reason, sequences of such graphs have very good properties as *expander graphs* (we refer to Kowalski (2019) for details on the link between spectral gap and expander properties of graphs). The question is then how to construct such sequences of graphs. In this direction, one has to mention the remarkable result of Friedman (2008):

Theorem 2.3. *Let $d \geqslant 3$ be an integer. For each $n \geqslant d+1$ such that nd is even, let G_n be a random graph chosen uniformly among d-regular graphs with n vertices. Then the sequence $(G_n)_{n\geqslant 1}$ is almost-Ramanujan*[(2)] *in the sense that, for any $\varepsilon > 0$,*

$$\lim_{n\to\infty} \mathbb{P}(\lambda(G_n) \geqslant 2\sqrt{d-1} + \varepsilon) = 0.$$

The notion of *random lift* is also very useful to construct almost-Ramanujan graphs. On the way of defining them, we also give a more general definition of Ramanujan graphs. Let G and H be undirected connected[(3)] graphs, with no loops and no multiple edges. A *covering map* π from H to G is a map from the set of vertices of H to the set of vertices of G such that, for every vertex h of H, π gives a bijection between the edges incident to h in H and those incident to $\pi(h)$ in G. When π is a covering map from H to G, the graph G is called the *base graph* and H is called a *covering graph* of G. Since the base graph G is connected, the cardinal of $\pi^{-1}(g)$ is the same for any vertex g of G. If this cardinal is equal to n, then H is called an *n-lift* of G. If H is a tree, it is the *universal cover* of G. In particular, the universal cover of any non-empty, connected, d-regular graph is the infinite d-regular tree $\mathbb{T}_d$. It is known that the spectrum of $\mathbb{T}_d$ is the interval $[-2\sqrt{d-1}, 2\sqrt{d-1}]$[(4)]. Therefore, for a d-regular graph, being a Ramanujan graph means that all eigenvalues, except $\pm d$, are contained inside the spectrum of its universal cover.

This led Greenberg (1995), in his thesis, to give a more general definition of a Ramanujan graph, not necessarily restricted to d-regular graphs.

Definition 2.4 (Ramanujan graph, general case)**.** A finite connected graph X is Ramanujan if the spectrum of its adjacency operator is contained in $[-\rho, \rho] \cup \{-\lambda_0, \lambda_0\}$, where λ_0 is the largest eigenvalue of the graph and ρ is the spectral radius of its universal cover.

(2) or weakly Ramanujan

(3) Covering maps can be defined in a more general framework but we only need the case of connected graphs.

(4) Its spectral measure is known as the Kesten–McKay distribution.

Let us describe a standard model, due to Amit and Linial (2002) for constructing *random lifts*. Given a base graph $G = (V_G, E_G)$ and an integer $n \geqslant 2$, to each vertex v of G we associate a set of n vertices $(v,1),\dots,(v,n)$. For each edge $\{u,v\}$ of G, we choose an orientation, say $e := (u,v)$, and a uniform permutation σ_e of $[n] := \{1,\dots,n\}$, independent of all other edges. Then, if the vertex set of H is $\{(u,i); u \in V_G, i \in [n]\}$ and the edge set is $\{\{(u,i),(v,\sigma_{(u,v)}(i))\}; \{u,v\} \in E_G, i \in [n]\}$, H is a *random n-lift* of G and its law is uniform over all possible n-lifts of G. Note that the choice of the orientations of the edges made for the construction does not change the distribution of the random lift H. Improving on Theorem 2.3, Bordenave (2020) showed that, under very general conditions on the base graph, the sequence of random n-lifts form a sequence of almost-Ramanujan graphs.

The model that we describe now is the main object studied by Bordenave and Collins (2019) and can be seen as a generalization of the notion of random lift; we will call it the *symmetric random permutation model*. Let X be a countable set. Let $\sigma_1,\dots,\sigma_d$ be d permutations of the set X. We consider $\ell^2(X)$ the Hilbert space spanned by the orthonormal basis $(\delta_x)_{x\in X}$. The identity operator on $\ell^2(X)$ is denoted by $\mathbf{1}$. A permutation σ_i acts naturally as a unitary operator S_i on $\ell^2(X)$ by $S_i(g)(x) = g(\sigma_i(x))$, for any $g \in \ell^2(X)$. Let $a_0, a_1, \dots, a_d$ be matrices of size $r \times r$. The main object of interest in Bordenave and Collins (2019) is the operator $A := a_0 \otimes \mathbf{1} + \sum_{i=1}^{d} a_i \otimes S_i$ acting on $\mathbb{C}^r \otimes \ell^2(X)$. When $X = [n]$, we denote by $\sigma_{1,n},\dots,\sigma_{d,n}$ the permutations of X, by $S_{1,n},\dots,S_{d,n}$ the corresponding operators, $\mathbf{1}^{(n)}$ the identity operator and

$$A_n := a_0 \otimes \mathbf{1}^{(n)} + \sum_{i=1}^{d} a_i \otimes S_{i,n}. \tag{1}$$

Two symmetry conditions are added, one on the matrices $a_1,\dots,a_d$ and one on the permutations $\sigma_{1,n},\dots,\sigma_{d,n}$.

Assumption 2.5 (Symmetric random permutation model)**.** We equip $[d]$ with the following involution: let $q \leqslant \frac{d}{2}$ be an integer; for any $i \in [q]$, set $i^* = i + q$, for $q+1 \leqslant i \leqslant 2q$, set $i^* = i - q$, and for $2q+1 \leqslant i \leqslant d$, set $i^* = i$. We assume that:

(H**a**) $a_0 = a_0^*$ and $\forall i \in \{1,\dots,d\}, a_{i^*} = (a_i)^*$.

(Hσ) The permutations $\{\sigma_{1,n},\dots,\sigma_{q,n}\} \cup \{\sigma_{2q+1,n},\dots,\sigma_{d,n}\}$ on $[n]$ are independent, $\{\sigma_{1,n},\dots,\sigma_{q,n}\}$ are uniformly distributed among the permutations of $[n]$ and $\{\sigma_{2q+1,n},\dots,\sigma_{d,n}\}$ are uniformly distributed among the *matchings*[(5)] of $[n]$ and for any $i \in [d]$, $\sigma_{i^*,n} = (\sigma_{i,n})^{-1}$.

[(5)] A matching (or pair matching) is a permutation for which all the cycles are of length 2, that is an involution without fixed point.

Let us explain how it can be seen as a generalization of the model of random lift. Assume that d is even, $q = d/2$, $a_0 = 0$ and the matrices $a_1, \dots, a_d$ are of the form $a_i = E_{u_i v_i}$, with $u_i, v_i \in [r]$. The base graph will be the graph G with vertex set $[r]$ and adjacency matrix $A_1 = \sum_{i=1}^{d} a_i$. Under Assumption (H**a**), it will be undirected, with $d/2$ edges. The graph H with vertex set $[n] \times [r]$ and edges of the form $\{(x, u_i), (\sigma_i(x), v_i)\}$ is a n-lift of G. If the permutations $\sigma_{1,n}, \dots, \sigma_{d,n}$ fulfill Assumption (Hσ), then the random lift we obtain has the same distribution as in the construction of Amit and Linial (2002).

Bordenave and Collins (2019) show that the $A_n, n \geqslant 1$, are the adjacency operators of an almost-Ramanujan sequence of weighted graphs, in a sense related to Definition 2.4 (we refer to Theorem 3.13 for a precise statement). But in parallel to this graph-theoretical motivation, the symmetric random permutation model is linked with asymptotic freeness properties of random permutations and we develop this point of view in the next section.

3. Freeness, asymptotic freeness and strong asymptotic freeness

In the eighties, Dan Voiculescu introduced the concept of *freeness* (or *free independence*) in the context of operator algebras and created the field of *free probability theory*. In the early nineties, he discovered that many models of random matrices were *asymptotically free*, leading to model elements in operator algebras through random matrices. Since then, there has been a constant interplay between free probability theory and *random matrix theory* (RMT). We will try to give the main lines of these fruitful interactions. Among many nice references on free probability theory, we have chosen to follow the recent book of Mingo and Speicher (2017) and the lecture notes of Speicher (2019).

3.1. The notion of freeness

Let us start with the definition of freeness.

Definition 3.1 (Freeness). Consider a unital algebra $\mathscr{A}$ over $\mathbb{C}$, equipped with a linear functional $\tau\colon \mathscr{A} \to \mathbb{C}$ such that $\tau(1) = 1$. The pair $(\mathscr{A}, \tau)$ is called a *non-commutative probability space*. Unital subalgebras $(\mathscr{A}_i)_{i \in I}$ are called *free* (or *freely independent*) in $(\mathscr{A}, \tau)$ if, for any $a_1, \dots, a_k$ such that $\forall j \in [k]$, $\tau(a_j) = 0, a_j \in \mathscr{A}_{i(j)}$ and $i(1) \neq i(2) \neq \cdots \neq i(k)$,

$$\tau(a_1 \cdots a_k) = 0.$$

An important example, which is particularly relevant in our context, is the following: