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EXPONENTIAL GROWTH RATES IN HYPERBOLIC GROUPS [*after* Koji Fujiwara and Zlil Sela]

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A classical result of Jørgensen and Thurston shows that the set of volumes of finite volume complete hyperbolic 3-manifolds is a well-ordered subset of the real numbers of order type ω^ω; moreover, each volume can only be attained by finitely many isometry types of hyperbolic 3-manifolds.

Fujiwara and Sela (2020) established a group-theoretic companion of this result: If Γ is a non-elementary hyperbolic group, then the set of exponential growth rates of Γ is well-ordered, the order type is at least ω^ω, and each growth rate can only be attained by finitely many finite generating sets (up to automorphisms).

In this talk, we outline this work of Fujiwara and Sela and discuss related results.

1. Main results

Geometric group theory provides a rich interaction between the Riemannian geometry of manifolds and the large-scale geometry of finitely generated groups. This bond is particularly strong in the presence of negative curvature and explains a variety of rigidity phenomena. The group-theoretic analogues of closed hyperbolic manifolds are hyperbolic groups; more generally, the group-theoretic analogues of finite volume complete hyperbolic manifolds are relatively hyperbolic groups.

The volume growth behaviour of Riemannian balls in the universal covering of a compact Riemannian manifold is the same as the growth behaviour of balls in Cayley graphs of the fundamental group. By definition, the exponential growth rates of finitely generated groups measure the exponential expansion rate of balls in Cayley graphs and thus are entropy-like invariants. While there is no direct connection between the volume of a hyperbolic manifold M and the exponential growth rates of $\pi_1(M)$, the results of Fujiwara and Sela (2020) show that certain sets of such values share fundamental structural similarities.

To state these results, for a finitely generated group Γ, we write $\operatorname{Exp}(\Gamma) \subset \mathbb{R}$ for the (countable) set of all exponential growth rates $\mathrm{e}(\Gamma, S)$ with respect to finite generating sets S of Γ. The automorphism group $\operatorname{Aut}(\Gamma)$ acts on the set $\operatorname{FG}(\Gamma)$ of all finite generating sets of Γ and $e(\Gamma, f(S)) = e(\Gamma, S)$ holds for all $S \in \operatorname{FG}(\Gamma)$ and all $f \in \operatorname{Aut}(\Gamma)$. More details on terminology and notation can be found in Appendix A.

Theorem 1.1 (well-orderedness; Fujiwara and Sela, 2020, Theorem 2.2). *If Γ is a hyperbolic group, then $\operatorname{Exp}(\Gamma)$ is well-ordered (with respect to the standard order on $\mathbb{R}$).*

Theorem 1.2 (finite ambiguity; Fujiwara and Sela, 2020, Theorem 3.1). *The set $\{S \in \operatorname{FG}(\Gamma) \mid e(\Gamma, S) = r\} / \operatorname{Aut}(\Gamma)$ is finite for every non-elementary hyperbolic group Γ and every $r \in \mathbb{R}$.*

Theorem 1.3 (growth ordinals; Fujiwara and Sela, 2020, Proposition 4.3). *Let Γ be a non-elementary hyperbolic group. Then the ordinal number $\operatorname{ord}_{\operatorname{Exp}}(\Gamma)$ associated with $\operatorname{Exp}(\Gamma)$ satisfies $\operatorname{ord}_{\operatorname{Exp}}(\Gamma) \geqslant \omega^\omega$.*

Moreover, Fujiwara and Sela (2020, Proposition 4.3) show that $\operatorname{ord}_{\operatorname{Exp}}(\Gamma) = \omega^\omega$ if epi-limit groups over Γ have a Krull dimension. In analogy with the case of hyperbolic 3-manifolds, they conjecture that $\operatorname{ord}_{\operatorname{Exp}}(\Gamma) = \omega^\omega$ holds for all non-elementary hyperbolic groups Γ (Fujiwara and Sela, 2020, Section 4) .

Example 1.4. If F is a finitely generated free group of rank at least 2, then limit groups over F have a Krull dimension (Louder, 2012). Hence, Theorems 1.1–1.3 show that $\operatorname{ord}_{\operatorname{Exp}}(F) = \omega^\omega$ and each value in $\operatorname{Exp}(F)$ is realised by only finitely many generating sets (up to automorphisms of F).

The key idea for the proofs of Theorems 1.1–1.3 is inspired by the proofs by Thurston and Jørgensen for the set of volumes of hyperbolic 3-manifolds and model theory: One passes from sequences of generating sets (of bounded size) of the given hyperbolic group Γ to a limit group over Γ with an associated finite generating set; *i.e.*, limit groups play the role of cusped manifolds. The main challenge is then to compute the exponential growth rate of this limiting object in terms of the exponential growth rates appearing in the original sequence.

Overview

Basics on hyperbolic groups, exponential growth rates, and well-ordered countable sets are recalled in Appendix A. We briefly explain the manifold context of the results above in Section 2, with a focus on hyperbolic and simplicial volume. Section 3 gives a proof outline of the main results. Finally, in Section 4, we mention applications and extensions of the main results.

2. Context: Volumes of manifolds and hyperbolicity

The results of Fujiwara and Sela (2020) are analogues of the behaviour of volumes of finite volume complete hyperbolic 3-manifolds. We recall this background in Section 2.1. The situation for simplicial volume is discussed in Section 2.2. In addition, we mention right-computability as a further structural property of "volume" sets (Section 2.3).

2.1. Hyperbolic volume

The structure and volumes of hyperbolic 3-manifolds was analysed in the breakthrough work of Jørgensen and Thurston.

Theorem 2.1 (volumes of hyperbolic 3-manifolds; Thurston, 1979, Chapter 6)**.** *The set*

$$\{\mathrm{vol}(M) \mid M \textit{ is a finite volume complete hyperbolic } 3\textit{-manifold}\}$$

is well-ordered (with respect to the standard order on $\mathbb{R}$*) and the associated ordinal is* ω^ω*. Moreover, every value arises only from finitely many isometry classes of finite volume hyperbolic* 3*-manifolds.*

We briefly summarise the main steps of the proof (Gromov, 1981); the key is to study the convergence of sequences of hyperbolic manifolds and to understand the role of hyperbolic manifolds with cusps as limits of such sequences:

1. Every sequence $(M_n)_{n\in\mathbb{N}}$ of complete hyperbolic 3-manifolds with uniformly bounded volume contains a subsequence that converges in a strong geometric sense to a finite volume complete hyperbolic 3-manifold M and $\lim_{n\to\infty} \mathrm{vol}(M_n) = \mathrm{vol}(M)$. Furthermore, for "non-trivial" such sequences, one can show that $\mathrm{vol}(M) > \mathrm{vol}(M_n)$ holds for all members M_n of the subsequence.

   This can be used to show that the set of hyperbolic volumes is well-ordered and that every value can only be obtained in finitely many ways.

2. Every finite volume complete hyperbolic 3-manifold with $k \in \mathbb{N}$ cusps can be obtained for each $p \in \{0, \dots, k\}$ as the limit of a sequence of finite volume complete hyperbolic 3-manifolds with exactly p cusps.

   This can be used to show that the volume ordinal is at least ω^k. Constructing hyperbolic 3-manifolds with arbitrarily large numbers of cusps thus shows that the volume ordinal is at least ω^ω. In combination with the first part, one can derive that the volume ordinal equals ω^ω.

In contrast, in higher dimensions, the set of volumes of finite volume complete hyperbolic manifolds leads to the ordinal ω. This follows from Wang's finiteness theorem and the unboundedness of hyperbolic volumes.

Theorem 2.2 (Wang's finiteness theorem; Wang, 1972). *Let $n \in \mathbb{N}_{\geqslant 4}$ and $v \in \mathbb{R}_{\geqslant 0}$. Then there exist only finitely many isometry classes of finite volume complete hyperbolic n-manifolds M with $\operatorname{vol}(M) \leqslant v$.*

2.2. Simplicial volume

Simplicial volume is a homotopy invariant of closed manifolds. For several geometrically relevant classes of Riemannian manifolds, the simplicial volume encodes topological rigidity properties of the Riemannian volume.

Definition 2.3 (simplicial volume; Gromov, 1982). The *simplicial volume* of an oriented closed connected manifold M is the ℓ^1-semi-norm of its (singular) $\mathbb{R}$-fundamental class:

$$\|M\| := \|[M]_{\mathbb{R}}\|_1 := \inf\Big\{\sum_{j=1}^{k} |a_j| \;\Big|\; \sum_{j=1}^{k} a_j \cdot \sigma_j \text{ is a singular } \mathbb{R}\text{-fundamental cycle of } M\Big\}$$

For genuine hyperbolic manifolds, the simplicial volume leads to the same ordering and finiteness behaviour as the hyperbolic volume (Section 2.1):

Example 2.4 (hyperbolic manifolds). If M is an oriented closed connected hyperbolic manifold of dimension n, then

$$\|M\| = \frac{\operatorname{vol}(M)}{v_n},$$

where $v_n \in \mathbb{R}_{>0}$ is the hyperbolic volume of ideal regular geodesic n-simplices in hyperbolic n-space (Benedetti and Petronio, 1992; Thurston, 1979). A similar relationship also holds in the complete finite volume case (Thurston, 1979; Fujiwara and Manning, 2011, Appendix A). In particular, this proportionality can be used to prove mapping degree estimates in terms of the hyperbolic volume for continuous maps between hyperbolic manifolds (Gromov, 1982).

Passing to the setting of fixed hyperbolic fundamental groups, we obtain:

Example 2.5 (hyperbolic fundamental group). Let Γ be a finitely presented group and let $n \in \mathbb{N}$. Then the set

$$\operatorname{SV}_\Gamma(n) := \{\|M\| \mid M \text{ is an oriented closed connected } n\text{-manifold with } \pi_1(M) \cong \Gamma\}$$

is a subset of $\{\|\alpha\|_1 \mid \alpha \in H_n(\Gamma;\mathbb{R}) \text{ is integral}\}$, where a class in $H_n(\Gamma;\mathbb{R})$ is *integral* if it lies in the image of the change of coefficients map $H_n(\Gamma;\mathbb{Z}) \to H_n(\Gamma;\mathbb{R})$ (Löh, 2023, Section 3.1).

If Γ is hyperbolic and $n \geqslant 2$, then $\|\cdot\|_1$ is a norm on $H_n(\Gamma;\mathbb{R})$ (by the results of Mineyev (2001) on bounded cohomology and the duality principle). In particular: The set $\operatorname{SV}_\Gamma(n) \subset \mathbb{R}$ is well-ordered and for $n \geqslant 4$ the ordinal associated with $\operatorname{SV}_\Gamma(n)$ is

▷ either 0 (if $H_n(\Gamma;\mathbb{R}) \cong 0$);

▷ or ω (if $H_n(\Gamma; \mathbb{R}) \not\cong 0$): In this case, normed Thom realisation shows that indeed infinitely many different values are realised (Löh, 2023, Section 3.1).

For $n \geqslant 4$, finite ambiguity breaks down in this general topological setting: If M is an oriented closed connected n-manifold, then for each $k \in \mathbb{N}$, the manifold M and the iterated connected sums $M_k := M \# (S^2 \times S^{n-2})^{\# k}$ have the same simplicial volume (Gromov, 1982) and isomorphic fundamental groups. However, the manifolds $M_0, M_1, \dots$ all have different homotopy types (as can be seen from the homology in degree 2).

2.3. Right-computability

In the previous discussion, we focussed on the order structure of volumes and exponential growth rates. Many real-valued invariants in geometric group theory and geometric topology also carry another, complementary, structure: They tend to have an intrinsic limit on their computational complexity. In particular, such a limit gives additional constraints on the possible sets of values.

Definition 2.6 (right-computable). A real number α is *right-computable* if the set $\{x \in \mathbb{Q} \mid x > \alpha\}$ is recursively enumerable.

For example, simplicial volumes of oriented closed connected manifolds are right-computable real numbers (Heuer and Löh, 2023). On the group-theoretic side, right-computability naturally arises for stable commutator length of recursively presented groups (Heuer, 2019) or L^2-Betti numbers of groups with controlled word problem (Löh and Uschold, 2022). Concerning exponential growth rates, we have the following:

Proposition 2.7 (right-computability of exponential growth rates). *There exists a Turing machine that*

▷ *given a finite presentation $\langle S \mid R \rangle$ and a finite set S' of words over $S \sqcup S^{-1}$,*

▷ *does*

- not *terminate if S' does not represent a generating set of the group Γ described by $\langle S \mid R \rangle$;*
- *terminate and return an enumeration of $\{x \in \mathbb{Q} \mid x > \mathrm{e}(\Gamma, S')\}$ if S' represents a generating set of Γ.*

Corollary 2.8. *Let Γ be a finitely presented group.*

1. *For every $S \in \mathrm{FG}(\Gamma)$, the real number $\mathrm{e}(\Gamma, S)$ is right-computable.*
2. *For every $r \in \mathbb{Q}$, the truncated set $\{S \in \mathrm{FG}(\Gamma) \mid \mathrm{e}(\Gamma, S) < r\}$ is recursively enumerable.*

Proofs of these observations are provided in Appendix B. In particular, such results could be used to give a crude a priori upper bound for $\mathrm{ord}_{\mathrm{Exp}}(\Gamma)$ by a "large" countable ordinal for all finitely presented groups Γ with well-ordered set $\mathrm{Exp}(\Gamma)$.