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STRONG FORCING AXIOMS AND THE CONTINUUM PROBLEM
[*after* Asperó's and Schindler's proof that $\mathbf{MM}^{++}$ implies Woodin's Axiom $(*)$]

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Introduction

This note addresses the continuum problem, taking advantage of the breakthrough mentioned in the subtitle, and relating it to many recent advances occurring in set theory.(1) We try to the best of our possibilities to make our presentation self-contained and accessible to a general mathematical audience.(2)

Let us start by stating Asperó's and Schindler's result:

Theorem 0.1 (Asperó and Schindler, 2021). *Assume* $\mathbf{MM}^{++}$ *holds. Then Woodin's axiom* $(*)$ *holds as well.*

We will address the following three questions:

- ▷ What is the axiom $\mathbf{MM}^{++}$?
- ▷ What is Woodin's axiom $(*)$?
- ▷ What is the bearing of Asperó's and Schindler's result on the continuum problem, and why their result is regarderd as a major breakthrough in the set theoretic community?

We give rightaway a spoiler of the type of answers we sketch for the above questions.

We have two major approaches to produce witnesses x of certain mathematical properties $P(x)$.

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(2)Surveys on the topic complementing this note are (among an ample list) Bagaria, 2005; Koellner, 2010; Venturi and Viale, 2023b; Woodin, 2001a,b.

A topological approach is exemplified by Baire's category theorem: given a compact Hausdorff topological space X one can find a "generic" point $x \in X$ satisfying a certain topological property $P(x)$ by showing that $P(x)$ can fail only on a "small" (more precisely meager) set of points of X.

An algebraic approach is exemplified by the construction of algebraic numbers: one takes a set of Diophantine equations $P(\vec{x})$ which are not jointly inconsistent, and builds abstractly a formal solution in the ring $\mathbb{Q}(\vec{x})/_{P(\vec{x})}$.

Duality theorems connect the algebraic point of view to the geometric one, for example Hilbert's Nullstellensatz relates solutions of irreducible sets of Diophantine equations to generic points of algebraic varieties.

We will outline that Woodin's axiom $(*)$ provides an "algebraic approach" to the construction of set theoretic witnesses for "elementary" set theoretic properties, $\mathbf{MM}^{++}$ a "geometric approach", and Asperó's and Schindler's result connects these two perspectives.

We plan to do this while gently introducing the reader to the fundamental concepts of set theory.

The note is structured as follows:

- ▷ 1 is a brief review of the basic results of set theory with a focus on its historical development and on the topological complexity of sets of reals witnessing the failure of the continuum hypothesis.
- ▷ In 2 we quote some of Gödel's thoughts on the continuum problem and on the ontology of mathematical entities.
- ▷ 3 gives a brief overview of (the use in mathematics of) large cardinal axioms.
- ▷ In 4 we introduce forcing axioms with a focus on their topological presentations, while giving a precise formulation of the axiom $\mathbf{MM}^{++}$. We also list some of the major undecidable problems which get a solution assuming this axiom, among which the continuum problem.
- ▷ 5 is a small interlude giving some insights on the forcing method, while relating it to the notions of sheaf and of Grothendieck topos.
- ▷ 6 revolves about the notion of algebraic closure. In particular we outline how Robinson's notion of model companionship gives the means to transfer the concept of "algebraic closure" developed for rings to a variety of other mathematical theories.
- ▷ 7 discusses what is the right language in which set theory should be axiomatized in order to unfold its "algebraic closure" properties.

- ▷ 8 relates Woodin's generic absoluteness results for second order number theory to properties of algebraic closure for the initial fragment of the universe of sets given by $H_{\aleph_1}$.

- ▷ 9 brings to light why Woodin's axiom $(*)$ can be regarded as an axiom of "algebraic closure" for the larger initial fragment of set theory given by $H_{\aleph_2}$. Putting everything together we conclude by showing why Asperó's and Schindler's result establish a natural correspondence between the geometric approach and the algebraic approach to forcing axioms.

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1. Basics of set theory

Set theory deals with the properties of sets (the "manageable" mathematical objects) and classes (the "not so manageable" entities).[3]

1.1. Axioms

The axioms of set theory can be split in three types (as is the case for many other mathematical theories):

- ▷ **Universal axioms** which establish properties valid for all sets;

- ▷ **Existence axioms** which establish the existence of certain sets;

- ▷ **Construction principles** which allow for the construction of new sets from ones which are already known to exist.

We present the axiomatization of set theory by Morse–Kelley MK with sets and classes. Its axioms are distributed in the three categories as follows:

Universal axioms

- ▷ **Extensionality:** Two classes (or sets) are equal if they have exactly the same elements.

[3] We refer the reader to Jech, 2003; Kunen, 1980; Monk, 1969 for a systematic treatment of the topic. The reader familiar with set theory can skim through or just skip this section.

- ▷ **Comprehension (a):** Every class (or set) is a subset of V, the (proper) class whose elements are exactly the sets.

  (a **proper class** is a class which is not a set, a **set** is a class which belongs to V).
- ▷ **Foundation:** There is no infinite sequence $\langle x_n : n \in \mathbb{N}\rangle$ of classes such that $x_{n+1} \in x_n$ for all n.

Existence axioms

- ▷ **Infinity:** $\emptyset$ and $\mathbb{N}$ are sets.

Weak construction principles

- ▷ **Union, Pair, Product:** If X, Y are sets, so are $X \cup Y$, $\{X, Y\}$, $X \times Y$.
- ▷ **Separation:** If P is a class and X is a set, $P \cap X$ is a set.

Strong construction principles

- ▷ **Comprehension (b):** For every property $\psi(x)$, $P_\psi = \{a \in V : \psi(a)\}$ is a class.
- ▷ **Replacement:** If F is a class function and $X \subseteq \text{dom}(F)$ is a set, the pointwise image $F[X]$ of X under F is a set.
- ▷ **Powerset:** If X is a set, so is the class $\mathscr{P}(X) = \{Y : Y \subseteq X\}$.
- ▷ **Global Choice:** For all classes $C = \{X_i : i \in I\}$ of non-empty sets X_i, $\prod_{i \in I} X_i$ (the family of functions F with domain I and such that $F(i) \in X_i$ for all $i \in I$) is non-empty.

Some comments:

- ▷ By Foundation V cannot be a set else $\langle x_n : n \in \mathbb{N}\rangle$ with each x_n constantly assigned to V defines a decreasing $\in$-chain.[4]
- ▷ Many of the objects of interest in mathematics are proper classes, for example the family of groups, or the family of topological spaces. More generally for a given (first order) theory T, the family of stuctures which satisfy the axioms of T is a proper class (and exists in view of Comprehension (b)). There are delicate ontological issues related to the notion of proper class, but they are foreign to almost all domains of mathematics, with the notable exceptions of category theory and set theory.

(4) *V is not a set* can also be proved without Foundation. Set theorists need foundation in order to infer that the notion of well-foundedness is an elementary set theoretic property (more precisely it is a provably Δ_1-property).

▷ It is convenient for natural numbers to distinguish their ordinal type (which confronts them according to which of these numbers "comes first") from their cardinal type (which assigns to each natural number n the family of sets which have exactly n elements). When dealing with arbitrary sets, their ordinal type may not be defined, while the cardinal type always is. Von Neumann devised a simple trick to represent the finite ordinal types. One can inductively define the natural number n as the set $\{0, \ldots, n-1\}$ (*i.e.* $0 = \emptyset, 1 = \{\emptyset\}, 2 = \{\emptyset, \{\emptyset\}\}, \ldots$).(5)

▷ Set theoretic construction principles are of two sorts: the simple (or weak) ones are for example those bringing from sets X, Y to sets $X \cup Y$, $\{X, Y\}$, $X \times Y$, or from set X and class P to the set $P \cap X$; the strong ones are the power-set axiom, the replacement axiom, and the axiom of choice. Let us discuss briefly the role of such axioms in the development of routine mathematics.

Weak construction principles The integers and rationals can be constructed from the naturals using only weak construction principles: $\mathbb{Z}$ can be seen as the subset of $\mathbb{N} \times \{0,1\}$ which assigns the positive integers to the ordered pairs with second coordinate 0 and the negative ones to those pairs with second coordinate 1 (paying attention to the double counting of 0 as $(0,0)$ and $(0,1)$); $\mathbb{Q}$ can be seen as the subset of $\mathbb{Z} \times (\mathbb{N} \setminus \{0\})$ given by ordered pairs which are coprime.

Powerset axiom In order to build the reals from the rationals, one needs this axiom: $\mathbb{R}$ is the subset of $\mathscr{P}(\mathbb{Q})$ given by Dedekind cuts.

Replacement axiom An adequate development of set theory requires it: consider the function F on $\mathbb{N}$ given by $F(0) = \mathbb{N}$, $F(n+1) = \mathscr{P}(F(n))$. Without replacement it cannot be proved that F (or even the image of F) is a set, it might only be a proper class.

Choice Choice also has a special status in ordinary mathematics, and many mathematicians feel uneasy about it. However Choice is unavoidable: it is essential in the proofs of the Hahn–Banach theorem, of the existence of a base for infinite-dimensional vector spaces, or of the existence of a maximal ideal on a ring,...Even the equivalence of sequential continuity and topological continuity for real valued functions requires it: if $f : \mathbb{R} \to \mathbb{R}$ is not continuous at x, there is $\varepsilon > 0$ such that for each n one can find x_n so that $|x_n - x| < 1/n$ and $|f(x_n) - f(x)| > \varepsilon$. The sequence $(x_n)_n$ is (and in most cases can only be) defined appealing to (countable) Choice.

(5)The transfinite ordinal types (or the Von Neumann ordinals) are those (possibly infinite) sets α which are linearly ordered by $\in$ and are transitive (*i.e.* such that when $x \in y \in \alpha$, we have that $x \in \alpha$ as well). The proper class of Von Neumann ordinals is linearly well-ordered by $\in$. One can check that the natural numbers are the finite Von Neumann ordinals and that $\mathbb{N}$ (the set of finite Von Neumann ordinals) is the first infinite Von Neumann ordinal.