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Ted CHINBURG & Robert GURALNICK & David HARBATER

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THE LOCAL LIFTING PROBLEM FOR ACTIONS OF FINITE GROUPS ON CURVES

BY Ted CHINBURG, Robert GURALNICK AND David HARBATER

ABSTRACT. – Let k be an algebraically closed field of characteristic $p > 0$. We study obstructions to lifting to characteristic 0 the faithful continuous action ϕ of a finite group G on $k[[t]]$. To each such ϕ a theorem of Katz and Gabber associates an action of G on a smooth projective curve Y over k. We say that the KGB obstruction of ϕ vanishes if G acts on a smooth projective curve X in characteristic 0 in such a way that X/H and Y/H have the same genus for all subgroups $H \subset G$. We determine for which G the KGB obstruction of every ϕ vanishes. We also consider analogous problems in which one requires only that an obstruction to lifting ϕ due to Bertin vanishes for some ϕ, or for all sufficiently ramified ϕ. These results provide evidence for the strengthening of Oort's lifting conjecture which is discussed in [8, Conj. 1.2].

RÉSUMÉ. – Soit k un corps algébriquement clos de caractéristique $p > 0$. Nous étudions les obstructions au relèvement en caractéristique 0 d'une action fidèle et continue ϕ d'un groupe fini G sur $k[[t]]$. Le théorème de Katz-Gabber associe à ϕ une action du groupe G sur une courbe projective Y lisse sur k. La KGB-obstruction de ϕ est dite nulle si G agit sur une courbe projective lisse X de caractéristique 0 avec égalité des genres de X/H et Y/H pour tout sous-groupe $H \subset G$. Nous déterminons les groupes G pour lesquels la KGB-obstruction s'annule pour toute action ϕ. Nous considérons également des situations analogues pour lesquelles il suffit d'annuler l'obstruction de Bertin à relever une action ϕ ou toutes actions ϕ suffisamment ramifiées. Ces résultats renforcent les convictions en faveur de la conjecture de Oort généralisée aux relèvements d'une action fidèle sur une courbe projective lisse ([8], Conj. 1.2).

1. Introduction

This paper concerns the problem of lifting actions of finite groups on curves from positive characteristic to characteristic 0. Let k be an algebraically closed field of characteristic $p > 0$, and let Γ be a finite group acting faithfully on a smooth projective curve Y over k. We will say this action *lifts to characteristic* 0 if there is a complete discrete valuation ring R having

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characteristic 0 and residue field k for which the following is true. There is an action of Γ on a smooth projective curve $\tilde{Y}$ over R for which there is a Γ-equivariant isomorphism between $k \otimes_R \tilde{Y}$ and Y.

We focus in this paper on the following local version of this problem. Let $\phi : G \to \mathrm{Aut}_k(k[[t]])$ be an injective homomorphism from a finite group G into the group of continuous automorphisms of the power series ring $k[[t]]$ over k. The existence of such a ϕ implies G is the semi-direct product of a cyclic group of order prime to p (the maximal tamely ramified quotient) by a normal p-subgroup (the wild inertia group). One says ϕ *lifts to characteristic* 0 if there is an R as above such that ϕ can be lifted to an embedding $\Phi : G \to \mathrm{Aut}_R(R[[t]])$ in the sense that $k \otimes_R \Phi = \phi$.

The local and global lifting problems are connected in the following way by a result of Bertin and Mézard [3]. For each wildly ramified closed point y of Y, fix an identification of the completion of the local ring of Y at y with $k[[t]]$, and let $\phi_y : \Gamma(y) \to \mathrm{Aut}_k(k[[t]])$ be the resulting action of the inertia group $\Gamma(y)$ of y on this completion. Then ϕ lifts to characteristic 0 if and only if each ϕ_y does.

In [8] we studied the global lifting problem. We defined Γ to be a (global) Oort group for k if every action of Γ on a smooth projective curve over k lifts to characteristic 0. This terminology arises from Oort's conjecture in [21, §I.7] that all cyclic groups Γ have this property, or equivalently that every connected cyclic cover lifts to characteristic 0. We showed in [8] that all groups Γ which are Oort groups for k must be on a certain list of finite groups that is recalled in Remark 1.4 below, and we conjectured this list was complete. Some results by various authors concerning Oort's conjecture and the generalization proposed in [8] are discussed after Remark 1.4 below.

In this paper we will focus on three local versions of the results in [8]. We will consider which finite groups G that are semi-direct products of a cyclic prime to p-group with a normal p-subgroup have the following properties for the field k.

1. If every local action $\phi : G \to \mathrm{Aut}_k(k[[t]])$ lifts to characteristic 0 we call G a *local Oort group for* k (as in [8]).
2. If every local action $\phi : G \to \mathrm{Aut}_k(k[[t]])$ that is sufficiently ramified lifts to characteristic 0, we will call G an *almost local Oort group for* k. More precisely, G is an almost local Oort group if there is an integer $N(G,k) \geq 0$ such that a local action ϕ lifts to characteristic 0 provided $t^{N(G,k)}$ divides $\phi(\sigma)(t) - t$ in $k[[t]]$ for all elements $\sigma \in G$ of p-power order.
3. If there is at least one local action $\phi : G \to \mathrm{Aut}_k(k[[t]])$ which lifts to characteristic 0 we will call G a *weak local Oort group for* k.

Our goal is to show that any G which has one of the three above properties must be on a certain list of groups associated to that property. In view of Oort's conjecture concerning cyclic groups we will ask to what extent these lists are complete.

The lists that we obtain will result from studying an obstruction to lifting ϕ that is due to Bertin [2], as well as from a refinement of this obstruction that we will call the KGB obstruction.

The *Bertin obstruction of* ϕ *vanishes* if there is a finite G-set S with non-trivial cyclic stabilizers such that the character χ_S of the action of G on S equals $-a_\phi$ on the non-trivial elements of G, where a_ϕ is the Artin character associated to ϕ. (For the definition of a_ϕ see [28, Chap. VI].) The condition on χ_S is thus that

$$\chi_S = m \cdot \mathrm{reg}_G - a_\phi \tag{1.1}$$

for some integer m, where reg_G is the character of the regular representation of G.

We will say that *Katz-Gabber-Bertin obstruction of ϕ vanishes*, or simply that the *KGB obstruction of ϕ vanishes*, if the following is true. There is a field K of characteristic 0 and a G cover $X \to X/G = \mathbb{P}^1_K$ of smooth geometrically irreducible projective curves over K such that

$$\mathrm{genus}(X/H) = \mathrm{genus}(Y/H)$$

for all subgroups H of G, where $Y \to Y/G = \mathbb{P}^1_k$ is the G-cover of smooth projective curves associated to ϕ by a theorem of Katz and Gabber (see [16]). Up to isomorphism, the Katz-Gabber cover $Y \to Y/G = \mathbb{P}^1_k$ is characterized by the fact that this G-cover is totally ramified over one point $\infty \in \mathbb{P}^1_k$, at most tamely ramified over another point $0 \in \mathbb{P}^1_k$, unramified off of $\{\infty, 0\}$, and the action of G on the completion $\hat{\mathcal{O}}_{Y,x}$ of the local ring of Y at the unique point x over ∞ corresponds to ϕ via a continuous k-algebra isomorphism between $\hat{\mathcal{O}}_{Y,x}$ and $k[[t]]$.

We prove in Theorem 4.2 that the Bertin obstruction vanishes if the KGB obstruction vanishes. In Appendix 2 we show that the KGB obstruction for ϕ need not vanish when the Bertin obstruction of ϕ does.

Definition 1.1. – Let G be a finite group which is the semi-direct product of a cyclic prime to p group by a normal p-subgroup. If the Bertin obstruction (resp. the KGB obstruction) vanishes for all ϕ then G will be called a *Bertin group* for k (resp. a *KGB group* for k). If this is true for all sufficiently ramified ϕ we call G an *almost Bertin group* for k (resp. an *almost KGB group* for k). Finally, if there is at least one ϕ for which the Bertin obstruction (resp. the KGB obstruction) vanishes, we will call G a *weak Bertin group* for k (resp. a *weak KGB group* for k).

Thus a local Oort group for k must be a KGB group for k, which must in turn be a Bertin group for k. One has a similar statement concerning almost local Oort groups and weak local Oort groups for k.

We can now state our main result concerning Bertin and KGB groups for k.

Theorem 1.2. – *Suppose G is a finite group which is a semi-direct product of a normal p-subgroup with a cyclic group of order prime to p. Let k be an algebraically closed field of characteristic p. Then G is a KGB group for k if and only if it is a Bertin group for k, and this is true exactly when G is isomorphic to a group of one of the following kinds:*

1. *A cyclic group.*
2. *The dihedral group D_{2p^n} of order $2p^n$ for some $n \geq 1$.*
3. *A_4 when $p = 2$.*
4. *A generalized quaternion group Q_{2^m} of order 2^m for some $m \geq 4$ when $p = 2$.*

Note that if $p = 2$ and $n = 1$ in item (2), D_4 is simply $\mathbb{Z}/2 \times \mathbb{Z}/2$.

By considering particular covers we showed in [8, Thm. 3.3, 4.4] that if G is a local Oort group for k then it must either be on the list given in Theorem 1.2 or else $p = 2$ and G is a semi-dihedral group of order at least 16. Theorem 1.2 shows that in fact, semi-dihedral groups are not local Oort groups in characteristic 2. Note also that Theorem 1.2 provides a necessary and sufficient condition for a group to be a Bertin group, which is equivalent to being a KGB group. Theorem 3.3 of [8] concerns only necessary conditions which must be satisfied by local Oort groups.

There are examples of particular actions ϕ for which the Bertin obstruction to lifting vanishes but the KGB obstruction does not (see Example B.2). Thus the fact that the Bertin and KGB groups turn out to be the same has to do with the requirement that the associated obstructions vanish for *all* such ϕ.

Pagot has shown in [23, Thm. 3] (see also [18]) that there are ϕ which have vanishing Bertin and KGB obstructions but which nonetheless do not lift to characteristic 0. Thus the latter obstructions are not sufficient to determine whether ϕ has a lift.

In view of Theorem 1.2, we asked the following question:

QUESTION 1.3. – *Is the set of groups listed in items [1]–[4] of Theorem 1.2 the set of all local Oort groups for algebraically closed fields k of characteristic p?*

Brewis and Wewers [7] have announced a proof that the answer to this question is negative because the generalized quaternion group of order 16 is not a local Oort group in characteristic 2.

REMARK 1.4. – Suppose the groups of type (1), (2) and (3) in Theorem 1.2 are all local Oort groups. It would then follow from [8, Thm. 2.4, Cor. 3.4, Thm. 4.5] that a cyclic by p group Γ is a global Oort group for k if and only if Γ is either cyclic, dihedral of order $2p^n$ for some n or (if $p = 2$) the alternating group A_4. This implication is not dependent on determining which generalized quaternion groups are local Oort groups in characteristic 2.

Oort's conjecture in [26] that cyclic groups are local and global Oort groups was shown for cyclic groups having a p-Sylow subgroup of order p (resp. p^2) by Oort, Sekiguchi and Suwa [26] (resp. by Green and Matignon [12]). Pagot showed (see [23] and [18]) that when $p = 2$, the Klein four group D_4 is a local and global Oort group. Bouw and Wewers have shown [4] that for all odd p, the dihedral group D_{2p} is a local and global Oort group, and they have announced a proof that when $p = 2$, A_4 is a local and global Oort group. In [5], Bouw, Wewers and Zapponi establish necessary and sufficient conditions for a given ϕ to lift to characteristic 0 whenever the p-Sylow subgroup of G has order p, regardless of whether G is a local Oort group.

The following is our main result concerning almost KGB groups and almost Bertin groups for k.

THEOREM 1.5. – *Suppose G is a finite group which is the semi-direct product of a cyclic group of order prime to p by a normal p-subgroup. Then G is an almost KGB group for k if and only if it is an almost Bertin group for k. The list of these groups consists of those appearing in Theorem 1.2 together with the groups $\mathrm{SL}_2(\mathbb{Z}/3)$ and Q_8 when $p = 2$.*