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VISCOUS BOUNDARY LAYERS IN HYPERBOLIC-PARABOLIC SYSTEMS WITH NEUMANN BOUNDARY CONDITIONS

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ABSTRACT. – We initiate the study of noncharacteristic boundary layers in hyperbolic-parabolic problems with Neumann boundary conditions. More generally, we study boundary layers with mixed Dirichlet-Neumann boundary conditions where the number of Dirichlet conditions is fewer than the number of hyperbolic characteristic modes entering the domain, that is, the number of boundary conditions needed to specify an outer hyperbolic solution. We have shown previously that this situation prevents the usual WKB approximation involving an outer solution with pure Dirichlet conditions. It also rules out the usual maximal estimates for the linearization of the hyperbolic-parabolic problem about the boundary layer.

Here we show that for linear, constant-coefficient, hyperbolic-parabolic problems one obtains a reduced hyperbolic problem satisfying Neumann or mixed Dirichlet-Neumann rather than Dirichlet boundary conditions. When this hyperbolic problem can be solved, a unique formal boundary-layer expansion can be constructed. In the extreme case of pure Neumann conditions and totally incoming characteristics, we carry out a full analysis of the quasilinear case, obtaining a boundary-layer approximation to all orders with a rigorous error analysis. As a corollary we characterize the small viscosity limit for this problem. The analysis shows that although the associated linearized hyperbolic and hyperbolic-parabolic problems do not satisfy the usual maximal estimates for Dirichlet conditions, they do satisfy analogous versions with losses.

RÉSUMÉ. – Nous initions l'étude des couches limites non caractéristiques de systèmes hyperboliques-paraboliques avec condition aux limites de Neumann. Plus généralement, nous étudions les couches limites avec condition aux limites de type mixte Dirichlet-Neumann, lorsque le nombre de conditions aux limites de Dirichlet est inférieur au nombre de modes caractéristiques rentrant dans le domaine, pour l'opérateur hyperbolique.

Dans le cas des systèmes linéaires à coefficients constants, nous obtenons un système hyperbolique limite avec des conditions aux limites de type Neumann ou Dirichlet-Neumann. Sous de bonnes hypothèses nous construisons des développements en couches limites BKW à tout ordre.

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Dans le cas extrême où tous les modes caractéristiques sont rentrants et avec des conditions de Neumann, nous traitons complètement le cas quasilinéaire, prouvant la convergence vers un problème hyperbolique limite avec des conditions de Neumann au bord. Les estimations maximales de stabilité obtenues pour les problèmes linéarisés sont plus faibles que celles typiques correspondant à des conditions de type Dirichlet.

1. Introduction

In the study of noncharacteristic boundary layers of hyperbolic-parabolic systems, physical applications motivate the inclusion of Neumann boundary conditions along with the usual Dirichlet boundary conditions that have traditionally been considered for such problems (see, e.g., [7, 20, 21] and rererences therein). In particular, as discussed in [17, 18, 11, 19], suction-induced drag reduction along an airfoil[(1)] is typically modeled by the compressible Navier-Stokes equations

$$(1.1)\qquad \begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0 \\ \partial_t(\rho u) + \mathrm{div}(\rho u^t u) + \nabla p = \varepsilon\mu\Delta u + \varepsilon(\mu+\eta)\nabla \mathrm{div} u \\ \partial_t(\rho E) + \mathrm{div}\big((\rho E + p)u\big) = \kappa\Delta T + \varepsilon\mu \mathrm{div}\big((u\cdot\nabla)u\big) + \varepsilon(\mu+\eta)\nabla(u\cdot \mathrm{div} u) \end{cases}$$

on an exterior domain Ω, with no-slip *suction-type* boundary conditions on the velocity, $u_T|_{\partial\Omega} = 0$, $u_\nu|_{\partial\Omega} = V(x) < 0$, and either prescribed or insulative boundary conditions on the temperature, $T|_{\partial\Omega} = T_{wall}(x)$ or $\partial_\nu T|_{\partial\Omega} = 0$.

The study of such mixed-type boundary layer problems was initiated in [11, 10] for certain combinations of Dirichlet and Neumann boundary conditions in the viscous problem. However, the ansatz used there, which assumes that the residual hyperbolic problem should have only Dirichlet boundary conditions, breaks down when there are too many Neumann conditions in the viscous problem - more precisely, when there are too few Dirichlet conditions, in the sense that the number of scalar Dirichlet conditions in the viscous problem is strictly less than the "correct" number of residual boundary conditions for the hyperbolic problem. In such cases, the construction in [11] of "$\mathscr{C}$-manifolds" of reachable states determining Dirichlet boundary conditions for the outer, hyperbolic solution fails, due to a lack of transversality, as a consequence of which (together with the low-frequency decomposition of [20]) the maximal linearized estimates used in [11, 10] to establish rigorous convergence may be shown to fail as well. As noted in [19], the case of (1.1) with incoming supersonic velocity falls into this category, so is not accessible by the techniques developed up to now.

Clearly, in such cases, a new analysis is required. Several questions arise, including:

(1) Does the hyperbolic-parabolic problem have a solution on a fixed time interval independent of ε?

(2) Is there a residual hyperbolic problem whose solution gives the small viscosity limit of solutions to the hyperbolic-parabolic problem? In particular, what are the correct residual hyperbolic boundary conditions? And, are these uniquely determined?

[(1)] See [22, 2], or NASA site `http://www.dfrc.nasa.gov/Gallery/photo/F-16XL2/index.html`.

(3) What are the maximal linearized estimates that we may expect in this context, both for the residual hyperbolic and full hyperbolic-parabolic problem?

In this paper, we answer these questions completely in the extreme case of pure Neumann boundary conditions and totally incoming hyperbolic characteristic modes, showing that there is a reduced hyperbolic problem with Neumann instead of Dirichlet conditions, and that in place of the standard Dirichlet-type linearized estimates for the reduced hyperbolic and full hyperbolic-parabolic systems, there hold modified versions with losses, sufficient to close a rigorous convergence argument. As a corollary we characterize the small viscosity limit for the quasilinear problem.

In the general, linear constant-coefficient case, we present two approaches to constructing a formal boundary-layer expansion to all orders of the solution to the hyperbolic-parabolic problem. In general the reduced hyperbolic (outer) problem features mixed Dirichlet-Neumann boundary conditions. In the pure Neumann case we prove that the exact and approximate solutions to the hyperbolic-parabolic problem are close when ε is small.

Our results motivate the further study of first-order hyperbolic initial-boundary-value problems with Neumann or mixed Neumann-Dirichlet boundary conditions. This is at first sight a counterintuitive problem, since the normal derivative on the boundary is not controlled by the usual hyperbolic solution theory, and it does not seem to have received much attention before now. We regard this as one of the most interesting aspects of the analysis.

1.1. Linear systems with Neumann boundary conditions

First we examine a linear problem for which the above questions have a positive, and rather simple, answer. Let us consider the parabolic boundary value problem on $\overline{\mathbb{R}}^{d+1}_+ := \{x = (x', x_d) = (x_0, x'', x_d) \in \mathbb{R}^{d+1} : x_d \geq 0\}$:

$$Lu = f + \varepsilon \Delta_x u \text{ in } \{x_d > 0\}, \tag{1.2}$$

$$\partial_d u_{|x_d=0} = 0, \tag{1.3}$$

$$u_{|t<0} = 0, \tag{1.4}$$

where L is a symmetric hyperbolic operator with constant coefficients

$$L = \partial_t + \sum_{j=1}^{d} A_j \partial_j, \ t = x_0$$

and $f \in H^\infty(\overline{\mathbb{R}}^{1+d}_+)$ with $f_{|t<0} = 0$. The $N \times N$ matrices A_j are constant (for now), and the boundary is noncharacteristic:

$$\det A_d \neq 0.$$

We look for an approximate solution of the form

$$u^\varepsilon(x) = u_0(x) + \varepsilon u_1(x, \frac{x_d}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x_d}{\varepsilon}) + \cdots$$

with the usual profiles

$$u_j(x, z) = \underline{u}_j(x) + u^*_j(x', z), \quad j \geq 1,$$

where $\underline{u}_j$ is an "outer" solution, and u^*_j is a boundary layer profile which goes to 0 as $z \to \infty$.

REMARK 1.1. – One could postulate a more general profile $u_0(x,z) = \underline{u}_0(x) + u_0^*(x',z)$ at level $j = 0$; however, the resulting ε^{-1} order profile equations $A_d\partial_z u_0^* - \partial_z^2 u_0^* = 0$, with boundary condition $\partial_z(u_0^*)_{|z=0} = 0$ would give then $\partial_z u_0^* \equiv 0$, recovering the assumption $u_0 = u_0(x)$.

The profile equation obtained at the order ε^0 is

$$Lu_0 + A_d\partial_z u_1 - \partial_z^2 u_1 = f,$$

which leads to the two equations for u_0 and u_1^*:

$$Lu_0 = f \tag{1.5}$$

and

$$A_d\partial_z u_1^* - \partial_z^2 u_1^* = 0. \tag{1.6}$$

The boundary condition (1.3) gives at the order ε^0:

$$(\partial_d u_0)_{|x_d=0} + (\partial_z u_1^*)_{|z=0} = 0.$$

Hence the solution to the boundary layer equation (1.6) is

$$u_1^*(x',z) = -e^{zA_d}\, A_d^{-1}\partial_d u_0(x',0). \tag{1.7}$$

It follows that u_1^* is decreasing at $+\infty$ if and only if $\partial_d u_0|_{x_d=0}$ lies in $\mathbb{E}_-(A_d)$, the negative eigenspace of A_d:

$$\partial_d u_{0|x_d=0} \in \mathbb{E}_-(A_d). \tag{1.8}$$

But u_0 satisfies $Lu_0 = f$; thus

$$\partial_d u_0 = -A_d^{-1}\sum_0^{d-1} A_j\partial_j u_0 + A_d^{-1} f$$

and the condition (1.8) is equivalent to

$$Hu_{0|x_d=0} \in A_d^{-1} f|_{x_d=0} + \mathbb{E}_-(A_d), \tag{1.9}$$

where H is the tangential operator $H := A_d^{-1}\sum_0^{d-1} A_j\partial_j$. So we are led to solve the mixed problem

$$Lu_0 = f \text{ in } \{x_d > 0\}, \tag{1.10}$$

$$Hu_{0|x_d=0} \in A_d^{-1} f_{|x_d=0} + \mathbb{E}_-(A_d), \tag{1.11}$$

$$u_{0|t<0} = 0. \tag{1.12}$$

(The boundary conditions may be rephrased via projections as described in Remark 1.4.)

To solve this problem introduce the unknown $v := Hu_0$, which is the solution of the symmetric hyperbolic problem with dissipative boundary conditions

$$Hv + \partial_d v = H(A_d^{-1} f) \text{ in } \{x_d > 0\}, \tag{1.13}$$

$$v_{|x_d=0} \in A_d^{-1} f_{|x_d=0} + \mathbb{E}_-(A_d), \tag{1.14}$$

$$v_{|t<0} = 0. \tag{1.15}$$