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STRONGLY INTERACTING BLOW UP BUBBLES FOR THE MASS CRITICAL NONLINEAR SCHRÖDINGER EQUATION

BY YVAN MARTEL AND PIERRE RAPHAËL

ABSTRACT. – We consider the mass critical two dimensional nonlinear Schrödinger equation

$$i\partial_t u + \Delta u + |u|^2 u = 0, \ \ t \in \mathbb{R}, \ x \in \mathbb{R}^2. \tag{NLS}$$

Let Q denote the positive ground state solution of $\Delta Q - Q + Q^3 = 0$. We construct a new class of multi-solitary wave solutions of (NLS) based on Q: given any integer $K \geq 2$, there exists a global (for $t > 0$) solution $u(t)$ that decomposes asymptotically into a sum of solitary waves centered at the vertices of a K-sided regular polygon and concentrating at a logarithmic rate as $t \to +\infty$, so that the solution blows up in infinite time with the rate

$$\|\nabla u(t)\|_{L^2} \sim |\log t| \ \text{ as } \ t \to +\infty.$$

Using the pseudo-conformal symmetry of the (NLS) flow, this yields the first example of solution $v(t)$ of (NLS) blowing up in finite time with a rate strictly above the pseudo-conformal one, namely,

$$\|\nabla v(t)\|_{L^2} \sim \left|\frac{\log|t|}{t}\right| \ \text{ as } \ t \uparrow 0.$$

Such a solution concentrates K bubbles at a point $x_0 \in \mathbb{R}^2$, that is $|v(t)|^2 \rightharpoonup K\|Q\|_{L^2}^2 \delta_{x_0}$ as $t \uparrow 0$. These special behaviors are due to strong interactions between the waves, in contrast with previous works on multi-solitary waves of (NLS) where interactions do not affect the global behavior of the waves.

RÉSUMÉ. – On considère l'équation de Schrödinger non linéaire critique pour la masse en dimension deux

$$i\partial_t u + \Delta u + |u|^2 u = 0, \ \ t \in \mathbb{R}, \ x \in \mathbb{R}^2. \tag{SNL}$$

Soit Q la solution positive et état fondamental de l'équation $\Delta Q - Q + Q^3 = 0$. On construit une nouvelle classe d'ondes solitaires multiples basées sur Q : étant donné un entier $K \geq 2$, il existe une solution globale (pour $t > 0$) $u(t)$ de (SNL) qui se décompose asymptotiquement en une somme d'ondes solitaires centrées sur les sommets d'un polygone régulier et qui se concentrent à un taux logarithmique quand $t \to +\infty$, de sorte que la solution explose en temps infini

$$\|\nabla u(t)\|_{L^2} \sim |\log t| \ \text{ quand } \ t \to +\infty.$$

Comme conséquence de la symétrie pseudo-conforme du flot de (SNL), on obtient le premier exemple d'une solution $v(t)$ de (SNL) qui explose en temps fini avec un taux strictement supérieur au taux

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pseudo-conforme

$$\|\nabla v(t)\|_{L^2} \sim \left|\frac{\log|t|}{t}\right| \quad \text{quand} \quad t \uparrow 0.$$

Cette solution concentre K bulles en un point $x_0 \in \mathbb{R}^2$, c'est-à-dire $|v(t)|^2 \rightharpoonup K\|Q\|_{L^2}^2\delta_{x_0}$ quand $t \uparrow 0$. Ces comportements particuliers sont dus aux interactions fortes entre les ondes solitaires, par opposition avec les résultats précédents sur les ondes solitaires multiples pour (SNL) où les interactions n'affectent pas le comportement global des ondes.

1. Introduction

1.1. General setting

We consider in this paper the mass critical two dimensional non linear Schrödinger equation (NLS)

$$i\partial_t u + \Delta u + |u|^2 u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^2. \tag{1.1}$$

It is well-known (see e.g., [7] and the references therein) that for any $u_0 \in H^1(\mathbb{R}^2)$, there exists a unique maximal solution $u \in \mathcal{C}((-T_\star, T^\star), H^1(\mathbb{R}^2))$ of (1.1) with $u(0) = u_0$. Moreover, the following blow up criterion holds

$$T^\star < +\infty \ \text{ implies } \ \lim_{t \uparrow T^\star} \|\nabla u(t)\|_{L^2} = +\infty. \tag{1.2}$$

The mass (i.e., the L^2 norm) and the energy E of the solution are conserved by the flow, where

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 - \frac{1}{4}\int_{\mathbb{R}^2} |u|^4.$$

From a variational argument, the unique (up to symmetry) ground state solution to

$$\Delta Q - Q + Q^3 = 0, \ \ Q \in H^1(\mathbb{R}^2), \ \ Q > 0, \ \ Q \text{ is radially symmetric},$$

attains the best constant C in the following Gagliardo-Nirenberg inequality

$$\forall u \in H^1(\mathbb{R}^2), \ \ \|u\|_{L^4}^4 \le C\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2 \tag{1.3}$$

(see [4, 54, 25]). As a consequence, one has

$$\forall u \in H^1(\mathbb{R}^2), \ \ E(u) \ge \frac{1}{2}\|\nabla u\|_{L^2}^2\left(1 - \frac{\|u\|_{L^2}^2}{\|Q\|_{L^2}^2}\right). \tag{1.4}$$

Together with the conservation of mass and energy and the blow up criterion (1.2), this implies the global existence of any solution with initial data $\|u_0\|_2 < \|Q\|_2$. Actually it is now known that in this case, the solution scatters i.e., behaves asymptotically in large time as a solution of the linear equation, see [19, 12] and references therein.

We also know that $\|u\|_{L^2} = \|Q\|_{L^2}$ corresponds to the mass threshold for global existence since the pseudo-conformal symmetry of the (NLS) equation

$$v(t,x) = \frac{1}{|t|}u\left(\frac{1}{|t|}, \frac{x}{|t|}\right)e^{-i\frac{|x|^2}{4|t|}} \tag{1.5}$$

applied to the solitary wave solution $u(t,x) = e^{it}Q(x)$ yields the existence of an explicit single bubble blow up solution $S(t)$ with minimal mass

$$S(t,x) = \frac{1}{|t|}Q\left(\frac{x}{|t|}\right)e^{-i\frac{|x|^2}{4|t|}}e^{\frac{i}{|t|}}, \quad \|S(t)\|_{L^2} = \|Q\|_{L^2}, \quad \|\nabla S(t)\|_{L^2} \underset{t\sim 0^-}{\sim} \frac{1}{|t|}. \tag{1.6}$$

We refer to [7] for more properties of the pseudo-conformal transform. From [35], minimal mass blow up solutions are *classified* in $H^1(\mathbb{R}^2)$:

$$\|u(t)\|_{L^2} = \|Q\|_{L^2} \text{ and } T^* < +\infty \text{ imply } u \equiv S \text{ up to the symmetries of the flow.}$$

Recall also the following well-known general sufficient criterion for finite time blow up: for initial data $u_0 \in \Sigma = H^1 \cap L^2(|x|^2dx)$, the virial identity

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^2}|x|^2|u(t,x)|^2dx = 16E(u_0) \tag{1.7}$$

implies blow up in finite time provided $E(u_0) < 0$ (by (1.4), this implies necessarily $\|u_0\|_{L^2} > \|Q\|_{L^2}$).

1.2. Single bubble blow up dynamics

We focus now on the case of mass slightly above the threshold, that is

$$\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha_0, \quad 0 < \alpha_0 \ll 1. \tag{1.8}$$

We first recall in this context that a large class of finite time blow up solutions was constructed in [6] (see also [22], [43]) as weak perturbation of the minimal mass solution $S(t)$. In particular, these solutions blow up with the pseudo-conformal blow up rate

$$\|\nabla u(t)\|_{L^2} \underset{t\sim T^*}{\sim} \frac{1}{T^*-t}. \tag{1.9}$$

Second, recall that the series of works [49, 37, 38, 52, 36, 40] provides a thorough study of the *stable* blow up dynamics under condition (1.8), corresponding to the so called *log-log* blow up regime

$$\|\nabla u(t)\|_{L^2} \underset{t\sim T^*}{\sim} c^*\sqrt{\frac{\log|\log(T^*-t)|}{T^*-t}}. \tag{1.10}$$

Third, it is proved in [43] (see also [22]) that solutions constructed in [6] are unstable and correspond in some sense to a threshold between the above log-log blow up and scattering.

Finally, recall that under (1.8), a universal gap on the blow up speed was proved in [52]: given a finite time blow up solution satisfying (1.8), either it blows up in the log-log regime (1.10), or it blows up faster than the pseudo-conformal rate

$$\|\nabla u(t)\|_{L^2} \gtrsim \frac{1}{T^*-t}.$$

(See also [1, 2].) However, the existence of solutions blowing up strictly faster than the conformal speed is a long lasting open problem, which is equivalent, by the pseudo-conformal symmetry (1.5), to the existence of global solutions blowing up in infinite time.