ISSN 0012-9593

quatrième série - tome 52 fascicule 6 novembre-décembre 2019

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Nguyen Viet DANG & Gabriel RIVIÈRE

Spectral analysis of Morse-Smale gradient flows

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 420 euros.
Abonnement avec supplément papier :
Europe : 551 €. Hors Europe : 620 € ($ 930). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Stéphane Seuret
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4ᵉ série, t. 52, 2019, p. 1403 à 1458

SPECTRAL ANALYSIS OF MORSE-SMALE GRADIENT FLOWS

BY NGUYEN VIET DANG AND GABRIEL RIVIÈRE

ABSTRACT. – On a smooth, compact and oriented manifold without boundary, we give a complete description of the correlation function of a Morse-Smale gradient flow satisfying a certain non-resonance assumption. This is done by analyzing precisely the spectrum of the generator of such a flow acting on certain anisotropic spaces of currents. In particular, we prove that this dynamical spectrum is given by linear combinations with integer coefficients of the Lyapunov exponents at the critical points of the Morse function. Via this spectral analysis and in analogy with Hodge-de Rham theory, we give an interpretation of the Morse complex as the image of the de Rham complex under the spectral projector on the kernel of the generator of the flow. This allows us to recover classical results from differential topology such as the Morse inequalities and Poincaré duality.

RÉSUMÉ. – Sur une variété lisse, compacte et orientée sans bord, nous donnons une description complète de la fonction de corrélation des flots de gradients Morse-Smale vérifiant certaines hypothèses de non-résonance. Ce résultat est obtenu en analysant précisément le spectre du générateur d'un tel flot agissant sur certains espaces de Sobolev anisotropes. Nous démontrons en particulier que ce spectre dynamique est donné par des combinaisons linéaires à coefficients entiers des exposants de Lyapunov aux points critiques de la fonction de Morse. Grâce à cette analyse spectrale et en analogie complète avec la théorie de Hodge-de Rham, nous interprétons le complexe de Morse comme l'image du complexe de de Rham par le projecteur sur le noyau du générateur du flot. Ceci nous permet de retrouver des résultats classiques de topologie différentielle comme les inégalités de Morse et la dualité de Poincaré.

1. Introduction

Consider a smooth ($\mathcal{C}^\infty$) flow $(\varphi^t)_{t\in\mathbb{R}}$ acting on a smooth, compact, oriented manifold M which has no boundary and which is of dimension $n \geq 1$. A natural question to raise is whether the limit

$$\lim_{t\to+\infty} \varphi^{-t*}(\psi)$$

 doi:10.24033/asens.2412

exists for any smooth function ψ defined on M. This is of course very unlikely to happen in general, and a natural setting where one may expect some convergence is the class of dynamical systems with hyperbolic behavior and for a nice enough reference measure. For instance, if φ^t is a topologically transitive Anosov flow [1] and if we study the weak limit with respect to a so-called Gibbs measure, it is known from the works of Bowen, Ruelle and Sinai that such a limit exists and is equal to the average of ψ with respect to the Gibbs measure (1) [45, 9]. If one is able to show that this equilibrium state exists, a second natural question to raise is: can one describe the fluctuations? For instance, what is the rate of convergence to this state?

These problems are naturally related to the study of the operator generating the flow:

$$\mathcal{L} : \psi \in \mathcal{C}^\infty(M) \mapsto -\frac{d}{dt}\left(\varphi^{-t*}(\psi)\right)|_{t=0} \in \mathcal{C}^\infty(M).$$

Note that, by duality, this operator acts on the space of distributions $\mathcal{D}'(M)$. In recent years, many progresses have been made in the study of such operators acting on suitable Banach spaces of distributions when the flow φ^t enjoys the Anosov property. In [36], Liverani defined Banach spaces of distributions with "anisotropic Hölder regularity" for which he could make a precise spectral analysis of $\mathcal{L}$ in the case of *contact Anosov flows*, and from which he could deduce that, for every $t \geq 0$ and for every ψ_1, ψ_2 in $\mathcal{C}^\infty(M)$,

$$C_{\psi_1,\psi_2}(t) := \int_M \varphi^{-t*}(\psi_1)\psi_2 d\mathrm{vol}_g = \int_M \psi_1 d\mathrm{vol}_g \int_M \psi_2 d\mathrm{vol}_g + \mathcal{O}_{\psi_1,\psi_2}(e^{-\Lambda t}), \tag{1}$$

where $\Lambda > 0$ is some fixed positive constant related to the spectral gap of $\mathcal{L}$ and where vol_g is the Riemannian volume. His construction was inspired by similar results for diffeomorphisms [7] and by a proof of Dolgopyat which holds in the 2-dimensional case [15]. Introducing Banach spaces inside $\mathcal{D}'(M)$ contrasts with earlier approaches to these questions where symbolic coding of Anosov flows was used to describe the weak convergence of $\varphi^{-t*}(\psi)$. For more general Anosov flows, Butterley and Liverani also showed how this direct approach allows to make a meromorphic extension for the Laplace transform of the correlation function $C_{\psi_1,\psi_2}(t)$ to the entire half plane [11]. This extended earlier works of Pollicott [40] and Ruelle [43] which were also based on the use of symbolic dynamics. Such poles (and their corresponding eigenstates) describe in some sense the fine structure of the long time dynamics and are often called *Pollicott-Ruelle resonances*. Pushing further this direct approach [25], Giulietti, Liverani and Pollicott extended this spectral analysis to anisotropic spaces of currents and they proved that, for any smooth Anosov flow, the Ruelle zeta function has a meromorphic extension to $\mathbb{C}$. In the case of Anosov geodesic flows satisfying certain pinching assumptions, they also showed that (1) also holds for the Bowen-Margulis measure (and not only with respect to the Riemannian volume). In parallel to this approach via spaces of anisotropic Hölder distributions, it was observed that the spectral analysis of Anosov flows can in fact be understood as a semiclassical problem which fits naturally in the theory of semiclassical resonances [31, 18]. Building on earlier works for Anosov diffeomorphisms by Baladi-Tsujii [3, 4] and Faure-Roy-Sjöstrand [20]

(1) Recall that a well-known example is the Liouville measure for the geodesic flow on a negatively curved manifold.

involving microlocal tools, this kind of approach to Pollicott-Ruelle resonances was developped for Anosov flows by Tsujii [50, 51], Faure-Sjöstrand [21], Faure-Tsujii [22, 23] and Dyatlov-Zworski [17]. We refer to the survey article of Gouëzel [26] for a recent account on these progresses.

Regarding the important steps made in the Anosov case, it is natural to understand to what extent these methods can be adapted to more general dynamical systems satisfying weaker chaotic features. A natural extension to consider is the class of Axiom A systems [48, 9]. In the case of nonsingular Axiom A flows, this was analyzed by Dyatlov and Guillarmou who showed that Pollicott-Ruelle resonances can be defined *locally* on a small neighborhood of any basic set of a given Axiom A flow with no critical points—see also [43, 3, 4, 27] for earlier results in the case of Axiom A diffeomorphisms. Here, we are aiming at analyzing the simplest class of Axiom A flows, namely Morse-Smale gradient flows. Yet, compared with the above references, our objective is to give a global description of the correlation function and not only in a neighborhood of the basic sets (here the critical points). Recall that gradient flows associated with a Morse function are also interesting because of their deep connections with differential topology which first appeared in the pioneering works of Thom [49] and Smale [46, 47]. Our microlocal approach to the properties of gradient flows will allow to give new (spectral) interpretations of some results of Laudenbach [34, 35] and Harvey-Lawson [29, 30] and to recover some classical facts from differential topology such as the finiteness of the Betti numbers, the Morse inequalities and Poincaré duality. Recall that an alternative spectral approach to Morse theory (based on Hodge theory) was introduced by Witten in [53]. For a more detailed exposition on these relations between dynamical systems, topology and spectral theory, we refer to the classical survey article of Bott [8].

2. Statement of the main results

2.1. Dynamical framework

We fix f to be a smooth ($\mathcal{C}^\infty$) Morse function, meaning that f has only finitely many critical points and that these points are non degenerate. We denote by $\text{Crit}(f)$ the set of critical points. For simplicity, we shall always assume that f is excellent in the sense that, given $a \neq b$ in $\text{Crit}(f)$, one has $f(a) \neq f(b)$. If we consider a smooth Riemannian metric g on M, we can define a vector field V_f as follows

$$\forall (x, v) \in TM,\ d_x f(v) = \langle V_f(x), v\rangle_{g(x)}. \tag{2}$$

This vector field generates a complete flow on M [35, Ch. 6] that we denote by φ_f^t. Given any point a in $\text{Crit}(f)$, we can define its stable (resp. unstable) manifold, i.e.,

$$W^{s/u}(a) := \left\{ x \in M :\ \lim_{t \to +/-\infty} \varphi_f^t(x) = a \right\}.$$

One can show that $W^s(a)$ (resp. $W^u(a)$) is an embedded submanifold in M of dimension $0 \leq r \leq n$ (resp. $n - r$) where r is the index of the critical point [52]. Note also that $W^u(a) \cap W^s(a) = \{a\}$. A remarkable property of these submanifolds is that *they form a*