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ON THE FIELDS OF DEFINITION OF HODGE LOCI

BY Bruno KLINGLER, Ania OTWINOWSKA
AND David URBANIK

ABSTRACT. – For $\mathbb{V} \to S$ a polarizable variation of Hodge structure defined over $\overline{\mathbb{Q}}$, the special subvarieties of S on which $\mathbb{V}$ admits exceptional Hodge tensors are conjectured to be defined over $\overline{\mathbb{Q}}$. We prove this conjecture for special subvarieties satisfying a simple monodromy condition, and illustrate this result for the universal family of smooth hypersurfaces of fixed degree in projective space. Using the same ideas, we moreover reduce the conjecture for special subvarieties of arbitrary dimension to the conjecture for special points.

RÉSUMÉ. – Étant donnée une variation polarisable de structures de Hodge $\mathbb{V} \to S$ définie sur $\overline{\mathbb{Q}}$, il est conjecturé que les sous-variétés spéciales de S le long desquelles $\mathbb{V}$ admet des tenseurs de Hodge exceptionnels sont définies sur $\overline{\mathbb{Q}}$. Nous démontrons cette conjecture pour les sous-variétés spéciales satisfaisant une condition simple de monodromie, et illustrons ce résultat dans le cas de la famille universelle des hypersurfaces lisses de degré fixé dans l'espace projectif. En utilisant les mêmes méthodes, nous réduisons la conjecture au cas particulier des points spéciaux.

1. Introduction

1.1. Hodge loci

The main objects of study in this article are Hodge loci. Let us start by recalling their definition in the geometric case, where their behavior is predicted by the Hodge conjecture.

Let $f : X \to S$ be a smooth projective morphism of smooth irreducible complex quasi-projective varieties and let i be a positive integer. The Betti and De Rham incarnations of the $2i$-th cohomology of the fibers of f give rise to a weight zero polarizable variation of Hodge structure $(\mathbb{V} := R^{2i} f_*^{\mathrm{an}}\mathbb{Z}(i), \mathcal{V} := R^{2i} f_*\Omega^\bullet_{X/S}, F^\bullet, \nabla)$ on S. Here $\mathbb{V}$ is the local system on the complex manifold S^{an} associated to S parametrising the $2i$-th Betti cohomology of the fibers of f; $\mathcal{V}$ is the corresponding algebraic vector bundle, endowed with its flat Gauß--Manin connection; and $F^\bullet$ is the Hodge filtration on $\mathcal{V}$ induced by the stupid filtration on the algebraic De Rham complex $\Omega^\bullet_{X/S}$. In this situation one defines the locus of exceptional

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Hodge classes $\mathrm{Hod}(\mathcal{V}) \subset \mathcal{V}^{\mathrm{an}}$ as the set of Hodge classes $\lambda \in F^0\mathcal{V}^{\mathrm{an}} \cap \mathbb{V}_{\mathbb{Q}}$ whose orbit under monodromy is infinite, and the Hodge locus $\mathrm{HL}(S, \mathbb{V})$ as its projection in S^{an}. Thus $\mathrm{HL}(S, \mathbb{V})$ is the subset of points s in S^{an} for which the Hodge structure $H^{2i}(X_s, \mathbb{Z}(i))$ admits more Hodge classes than the very general fiber $H^{2i}(X_{s'}, \mathbb{Z}(i))$.

According to the Hodge conjecture each $\lambda \in \mathrm{Hod}(\mathcal{V})$ should be the cycle class of an exceptional algebraic cycle in the corresponding fiber of f. As algebraic subvarieties of the fibers are parametrised by a common Hilbert scheme, the Hodge conjecture and an easy countability argument imply the following (as noticed by Weil in [20], where he asks for an unconditional proof):

($\star$) The locus of Hodge classes $\mathrm{Hod}(\mathcal{V})$ is a countable union of closed irreducible algebraic subvarieties of $\mathcal{V}$. The restriction of f to any such subvariety of $\mathcal{V}$ is finite over its image. In particular the Hodge locus $\mathrm{HL}(S, \mathbb{V})$ is a countable union of closed irreducible algebraic subvarieties of S.

More generally let $(\mathbb{V}, \mathcal{V}, F^\bullet, \nabla)$ be any polarizable variation of $\mathbb{Z}$-Hodge structure ($\mathbb{Z}$VHS) on a smooth complex irreducible algebraic variety S. Thus $\mathbb{V}$ is a finite rank $\mathbb{Z}_{S^{\mathrm{an}}}$-local system on the complex manifold S^{an}; and $(\mathcal{V}, F^\bullet, \nabla)$ is the unique regular algebraic module with integrable connection on S whose analytification is $\mathbb{V} \otimes_{\mathbb{Z}_{S^{\mathrm{an}}}} \mathcal{O}_{S^{\mathrm{an}}}$ endowed with its Hodge filtration $F^\bullet$ and the holomorphic flat connection ∇^{an} defined by $\mathbb{V}$, see [14, (4.13)]). We will abbreviate the $\mathbb{Z}$VHS $(\mathbb{V}, \mathcal{V}, F^\bullet, \nabla)$ simply by $\mathbb{V}$. If we define the locus of exceptional Hodge classes $\mathrm{Hod}(\mathcal{V}) \subset \mathcal{V}$ and the Hodge locus $\mathrm{HL}(S, \mathbb{V}) \subset S$ as in the geometric case, Cattani, Deligne and Kaplan [6] proved a vast generalization of Weil's expectation:

Theorem 1.1 (Cattani-Deligne-Kaplan). – *Let $\mathbb{V}$ be a polarizable $\mathbb{Z}$VHS on a smooth complex quasi-projective variety S. Then ($\star$) holds true.*

From now on we do not distinguish a complex algebraic variety X from its associated complex analytic space X^{an}, the meaning being clear from the context. It will be convenient for us to work in the following more general tensorial setting. Let $\mathbb{V}^\otimes$ be the infinite direct sum of $\mathbb{Z}$VHS $\bigoplus_{a,b\in\mathbb{N}} \mathbb{V}^{\otimes a} \otimes (\mathbb{V}^\vee)^{\otimes b}$, where $\mathbb{V}^\vee$ denotes the $\mathbb{Z}$VHS dual to $\mathbb{V}$; and let $(\mathcal{V}^\otimes, F^\bullet)$ be the corresponding filtered algebraic vector bundle of infinite rank. We denote by $\mathrm{Hod}(\mathcal{V}^\otimes) \subset \mathcal{V}^\otimes$ and $\mathrm{HL}(S, \mathbb{V}^\otimes) \subset S$ the corresponding locus of Hodge tensors and the tensorial Hodge locus respectively. Thus $\mathrm{HL}(S, \mathbb{V}^\otimes)$ is the subset of points s in S^{an} for which the Hodge structure $\mathbb{V}_s$ admits more Hodge *tensors* than the very general fiber $\mathbb{V}_{s'}$, equivalently where the Mumford-Tate group $\mathbf{G}_s$ of $\mathbb{V}_s$ is not of maximal dimension. Theorem 1.1 says that $\mathrm{Hod}(\mathcal{V}^\otimes)$ and $\mathrm{HL}(S, \mathbb{V}^\otimes)$ are countable unions of closed irreducible subvarieties of $\mathcal{V}^\otimes$ and S respectively, called *the special subvarieties of $\mathcal{V}^\otimes$ and S for* $\mathbb{V}$. We refer to [4] for a simplified proof of the statement for $\mathrm{HL}(S, \mathbb{V}^\otimes)$ using o-minimal geometry.

1.2. Fields of definition of Hodge loci

The question we attack in this paper is the relation between the field of definition of the $\mathbb{Z}$VHS $\mathbb{V}$ and the fields of definition of the corresponding special subvarieties.

Once again the geometric case provides us with a motivation and a heuristic. Suppose that $f : X \to S$ is defined over a number field $L \subset \mathbb{C}$. In that case one easily checks, refining Weil's argument, that the Hodge conjecture implies, in addition to $(\star)$:

$(\star\star)$
(a) each irreducible component of $\mathrm{Hod}(\mathcal{V})$, respectively $\mathrm{HL}(S, \mathbb{V})$, is defined over a finite extension of L;
(b) each of the finitely many $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$-conjugates of such a component is again an irreducible component of $\mathrm{Hod}(\mathcal{V})$, respectively $\mathrm{HL}(S, \mathbb{V})$.

REMARK 1.2. – Of course $(\star\star)$ for $\mathrm{Hod}(\mathcal{V})$ implies $(\star\star)$ for $\mathrm{HL}(S, \mathbb{V})$, and is a priori strictly stronger.

REMARK 1.3. – The full Hodge conjecture is not needed to deduce $(\star\star)$. As proven by Voisin [18, Lemma 1.4], the property $(\star\star)$ for $\mathrm{Hod}(\mathcal{V})$ is equivalent to the conjecture that Hodge classes in the fibers of f are (de Rham) absolute Hodge classes. We won't use the notion of absolute Hodge classes in this article and refer the interested reader to [7] for a survey. Our methods, being primarily concerned with the geometric properties of the special subvarieties themselves, say little directly about Hodge classes.

Let us now turn to general $\mathbb{Z}$VHS.

DEFINITION 1.4. – *We say that a $\mathbb{Z}$VHS $\mathbb{V}$ is* defined over a number field $L \subset \mathbb{C}$ *if S, $\mathcal{V}$, $F^\bullet$ and ∇ are defined over L: $S = S_L \otimes_L \mathbb{C}$, $\mathcal{V} = \mathcal{V}_L \otimes_L \mathbb{C}$, $F^\bullet\mathcal{V} = (F_L^\bullet \mathcal{V}_L) \otimes_L \mathbb{C}$ and $\nabla = \nabla_L \otimes_L \mathbb{C}$ with the obvious compatibilities.*

In the same way the property $(\star)$, which is implied by the Hodge conjecture in the geometric case, was proven to be true for a general $\mathbb{Z}$VHS, we expect the property $(\star\star)$, which is implied by the Hodge conjecture in the geometric case, to hold true for any $\mathbb{Z}$VHS $\mathbb{V}$, namely:

CONJECTURE 1.5. – *Let $\mathbb{V}$ be a $\mathbb{Z}$VHS defined over a number field $L \subset \mathbb{C}$. Then:*

(a) *any special subvariety of $\mathcal{V}^\otimes$ (resp. of S) for $\mathbb{V}$ is defined over a finite extension of L;*
(b) *any of the finitely many* $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$*-conjugates of a special subvariety of $\mathcal{V}^\otimes$ (resp. of S) for $\mathbb{V}$ is a special subvariety of $\mathcal{V}^\otimes$ (resp. of S) for $\mathbb{V}$.*

REMARK 1.6. – Simpson's non-abelian period conjecture [15, "Standard conjecture," p. 372] predicts that any $\mathbb{Z}$VHS defined over a number field $L \subset \mathbb{C}$ ought to be motivic: there should exist a $\overline{\mathbb{Q}}$-Zariski-open subset $U \subset S$ such that the restriction of $\mathbb{V}$ to U is a direct factor of a geometric $\mathbb{Z}$VHS on U. Thus Conjecture 1.5 would follow from Simpson's "standard conjecture" and $(\star\star)$ in the geometric case. Of course Simpson's standard conjecture seems unreachable with current techniques.

Let us mention the few results in the direction of Conjecture 1.5 we are aware of:

Suppose we are in the geometric situation of a morphism $f : X \to S$ defined over $\mathbb{Q}$. In [18, Theor. 0.6] (see also [19, Theor. 7.8]), Voisin proves the following: