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DEFORMATION QUANTIZATION

by **Alan WEINSTEIN**

1. INTRODUCTION

Quantum mechanics is often distinguished from classical mechanics by a statement to the effect that the observables in quantum mechanics, unlike those in classical mechanics, do not commute with one another. Yet classical mechanics is meant to give a description (with less precision) of the same physical world as is described by quantum mechanics. One mathematical transcription of this *correspondence principle* is the that there should be a family of (associative) algebras $\mathcal{A}_\hbar$ depending nicely in some sense upon a real parameter $\hbar$ such that $\mathcal{A}_0$ is the algebra of observables for classical mechanics, while $\mathcal{A}_\hbar$ is the algebra of observables for quantum mechanics. Here, $\hbar$ is the numerical value of Planck's constant when it is expressed in a unit of action characteristic of a class of systems under consideration. (This formulation avoids the paradox that we consider the limit $\hbar \to 0$ even though Planck's constant is a *fixed* physical magnitude.)

The first order (in $\hbar$) deviation of the quantum multiplication from the classical one is to be given by the Poisson bracket of classical observables. This idea goes back to Dirac [Di], who emphasized the analogies between classical Poisson brackets and quantum commutators. It played an important role in much of Berezin's work [Be1][Be2] on quantization.

Although the terminology and much of the inspiration comes from physics, noncommutative deformations of commutative algebras have also played a role of increasing importance in mathematics itself, especially since the advent of quantum groups about 15 years ago.

In the theory of *formal deformation quantization*, the "family of algebras $\mathcal{A}_\hbar$" is in fact a family $\star_\hbar$ of associative multiplications on a fixed complex vector space $\mathcal{A}$. More precisely, this family is given by a sequence of bilinear mappings

$B_j : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ for $j = 0, 1, \ldots$ so that

$$a \star_\hbar b = \sum B_j(a,b)\hbar^j.$$

The condition for associativity of the product is that

$$\sum_{j+k=n} B_j(a, B_k(b,c)) = \sum_{j+k=n} B_j(B_k(a,b),c)$$

for $n = 0, 1, 2, \cdots$.

The problem of formal deformation quantization is to classify such families up to equivalence, where an equivalence between formal deformations $\mathbf{B} = B_0, B_1, \ldots$ and $\mathbf{B}' = B_0', B_1', \ldots$ is, intuitively speaking, a formal family $G_{(\hbar)} : \mathcal{A} \to \mathcal{A}$ of maps such that $G_{(\hbar)}(a \star_\hbar b) = G_{(\hbar)}(a) \star_\hbar' G_{(\hbar)}(b)$. More precisely, such a family is given by a sequence $\mathbf{G} = G_0, G_1, \ldots$ of linear maps from $\mathcal{A}$ to $\mathcal{A}$ which satisfy the conditions

$$\sum_{j+k+r=n} B_r(G_j(a), G_k(b)) = \sum_{r+s=n} G_s(B_r'(a,b))$$

for $n = 0, 1, 2, \ldots$.

It is often useful to think of the deformation quantization as giving an associative algebra structure on the space $\mathcal{A}[[\hbar]]$ of formal power series with coefficients in $\mathcal{A}$ and an equivalence as giving an isomorphism between such algebras.

In attempting to solve the existence problem recursively for the B_j's, one finds at each stage an equation of the form $\delta B_j = F_j$, where F is a quadratic expression in the terms determined previously; a similar equation arises for each G_j in the equivalence problem. The operator δ goes from bilinear to trilinear (or linear to bilinear) $\mathcal{A}$-valued functionals on $\mathcal{A}$ and is precisely the coboundary operator for Hochschild cohomology with values in $\mathcal{A}$ of the algebra $\mathcal{A}$ with multiplication given by B_0. (In the equivalence problem, one normally assumes the product $\star_0$ as given, so that $B_0 = B_0'$, and G_0 is assumed to be the identity.) This cohomological approach to the deformation of algebras was established in the 1960's by Gerstenhaber [Ge].

A program to apply the methods of Gerstenhaber to algebras of interest in classical and quantum mechanics was laid out in 1978 by Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer [BFFLS]. (Another survey of the current state of the art may be found in [FS].) The aim of this program has been to develop as much as possible of quantum mechanics in terms of the deformed algebra structures, without using the customary representations in Hilbert spaces. Here, $\mathcal{A}$ is taken to be the space $C^\infty(M)$ of smooth complex-valued functions on a manifold M which represents the classical phase space. The undeformed product $\star_0$ (i.e. B_0) is taken to be the usual pointwise multiplication, so that $(\mathcal{A}, \star_0)$ is the algebra of classical observables. Next, following Dirac, it is assumed that the "limit" $\lim_{\hbar\to 0}[(a \star_\hbar b - b \star_\hbar a)/i\hbar]$ (i.e. $B_1(a,b) - B_1(b,a)/i\hbar$ is equal to a given classical Poisson bracket $\{a,b\}$ on $\mathcal{A}$. This bracket should be a *Poisson structure* in the sense that it satisfies the axioms of a Lie algebra together with the Leibniz identity $\{ab, c\} = \{a,c\}b + a\{b,c\}$. In this context, a formal deformation $\mathbf{B} = B_0, B_1, \ldots$ is called a $\star$-product (or star-product) if each of the bilinear maps B_j is a differential operator in each of its arguments, annihilating the constant functions when $j \geq 1$. These conditions make the $\star$-product local and insure that the constant function 1 remains as the unit element. Occasionally, the parity condition $B_j(a,b) = (-1)^j \overline{B_j(b,a)}$ is also imposed.

From here on, we will use the terms "$\star$-product" and "(deformation) quantization" interchangeably.

Among the Poisson manifolds (manifolds equipped with Poisson structure), the symplectic manifolds are of particular interest. We recall that a symplectic manifold is a manifold M equipped with a closed non-degenerate 2-form. According to Darboux's Theorem, such a manifold is always locally isomorphic to $\mathbf{R}^{2n}$ equipped with the symplectic form expressed in coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ as $\sum_i dq_i \wedge dp_i$. The Poisson structure

$$\{a,b\} = \sum_j \left(\frac{\partial a}{\partial q_j}\frac{\partial b}{\partial p_j} - \frac{\partial a}{\partial p_j}\frac{\partial b}{\partial q_j} \right)$$

is invariant under all diffeomorphisms preserving the symplectic form, so there is a well-defined Poisson structure on any symplectic manifold. Non-symplectic manifolds arise for instance as quotients of symplectic manifolds by symmetry groups and as the classical limits of quantum groups.

The fundamental example of a $\star$-product is the *Moyal-Weyl product* on $\mathbf{R}^{2n}$ with the Poisson structure just described. It comes from the composition of operators on $C^\infty(\mathbf{R}^n)$ via Weyl's identification [Wy] of such operators with functions on $\mathbf{R}^{2n}$, and was used by Moyal [My] to study quantum statistical mechanics from the viewpoint of classical phase space. The term B_1 in the formal series for this product is just $i/2$ times the "Poisson operator" $(a,b) \mapsto \{a,b\}$, and the full series is essentially the exponential of B_1. We will define the "powers" of the Poisson operator which enter in this series in a slightly more general setting. Let V be a vector space, and let π be a skew-symmetric bilinear functional on V^*. The formula $\{a,b\} = \pi(da,db)$ defines a Poisson structure on V. Associated to the bilinear operator π is a unique differential operator $\Pi : C^\infty(V \times V) \to C^\infty(V \times V)$ with constant coefficients for which $\{a,b\} = \Delta^*\Pi(a \otimes b)$; here, $a \otimes b$ is the function $(y,z) \mapsto a(y)b(z)$, and $\Delta^* : C^\infty(V \times V) \to C^\infty(V)$ is restriction to the diagonal. Now we define the Moyal-Weyl product on V by

$$a \star_\hbar b = \Delta^* \exp(i\hbar\Pi/2)(a \otimes b).$$

The space $C^\infty(V)[[\hbar]]$ with this product will be called the *Weyl algebra* of V and denoted by $W(V)$.

If $(x_1, \ldots, x_m)$ are linear coordinates on V, then the Poisson brackets $\{x_r, x_s\}$ are constants π_{rs} (the components of π), and the operator B_j in the expansion of the Moyal-Weyl product is

$$(1) \qquad B_j(a,b)(x) = \frac{1}{j!}\left(\frac{i}{2}\sum_{r,s}\pi_{rs}\frac{\partial}{\partial y_r}\frac{\partial}{\partial z_s})^j(a(y)b(z))\right)_{|y=z=x}.$$

On a general Poisson manifold, the Leibniz identity implies that the Poisson bracket is given by a skew-symmetric contravariant tensor (or "bivector") field π, called the Poisson tensor, via the formula $\{a,b\} = \pi(da,db)$. If the rank of the tensor π (i.e. the rank of the matrix function $\pi_{rs}(x) = \{x_r, x_s\}$ which represents it in local coordinates, or the rank of the corresponding mapping from 1-forms to vectors) is constant, then by a theorem of Lie [L] the Poisson manifold is locally isomorphic to a vector space with constant Poisson structure. Hence such Poisson manifolds, which are called *regular*, are always *locally* deformation quantizable; the problem is to patch together the local deformations to produce a global $\star$-product.

There is one case in which the patching together of local quantizations is easy. The Moyal-Weyl product on a vector space V with constant Poisson structure is invariant under all the affine automorphisms of V, since the notion of "operator