Bulletin

de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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Tome 135
Fascicule 4

2007

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Publié avec le concours du Centre national de la recherche scientifique

pages 1-

Bull. Soc. math. France
135 (4), 2007, p. 565–

THE TANGENT COMPLEX TO THE BLOCH-SUSLIN COMPLEX

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ABSTRACT. — Motivated by a renewed interest for the "additive dilogarithm" appeared recently, the purpose of this paper is to complete calculations on the tangent complex to the Bloch-Suslin complex, initiated a long time ago and which were motivated at the time by scissors congruence of polyedra and homology of SL_2. The tangent complex to the trilogarithmic complex of Goncharov is also considered.

RÉSUMÉ (*Le complexe tangent au complexe de Bloch-Suslin*). — À la suite de travaux récents sur le « dilogarithme additif », on se propose de compléter une étude du complexe tangent au complexe de Bloch-Suslin, initiée il y a plus de vingt ans en rapport avec le troisième problème de Hilbert et l'homologie de SL_2. On considère aussi le complexe tangent au complexe trilogarithmique de Goncharov.

1. Introduction

This paper could have been also entitled: "*Infinitesimal algebra of the Abel 5-term relation of the dilogarithm*". Our point of view is rather elementary: we complete calculations initiated a long time ago in [6], which were motivated by scissors congruences of polyedra in Euclidean three space. The subject has attracted recently new interest (see specially for another less naive point

Texte reçu le 7 juin 2007, accepté le 21 février 2008

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2000 Mathematics Subject Classification. — 11G55, 14F42, 19E15.

Key words and phrases. — Bloch-Suslin complex, additive dilogarithm, tangent functors.

of view, the work of Bloch-Esnault in [3]) and is related to many questions, namely (here k is a field)

- Homology of $\mathrm{SL}_2(k)$ with coefficients in the adjoint action and its exterior powers over $\mathbb{Z}$ [6, 12],
- Relative algebraic K-theory and relative cyclic homology of dual numbers $k[\varepsilon]$ [7, 11],
- Scissors congruences of polyedra in Euclidean spaces [9, 17, 5],
- Infinitesimal polylogarithms [8, 13],
- Cycles with modulus and additive polylogarithms [4, 3, 27, 26, 28],
- The possible existence of a theory of mixed Tate motives over dual numbers [16, 17, 3],
- Tangent spaces to spaces of algebraic cycles and formal deformations of Chow groups [20, 21].

Let the Bloch-Suslin complex of a field be the cohomological complex in degree 1 and 2

$$D : \mathcal{B}(k) \to \bigwedge^2 k^\times,$$

where $\mathcal{B}(k)$ is the commutative group with generators symbols $\{a\}$, where $a \in k^\times$ and $a \neq 1$, submitted to the five terms relation

$$\{a\} - \{a'\} + \left\{\frac{a'}{a}\right\} - \left\{\frac{1-a'}{1-a}\right\} + \left\{\frac{a(1-a')}{a'(1-a)}\right\} = 0,$$

and

$$D(\{a\}) = a \wedge (1-a).$$

When k is algebraically closed of characteristic 0, this complex can be inserted into several exact sequences (see Suslin [31] for finer results): the Bloch-Wigner sequence [10]

$$0 \to \mu(k) \to H_3(\mathrm{SL}_2(k), \mathbb{Z}) \to \mathcal{B}(k) \to \bigwedge^2 k^\times \to H_2(\mathrm{SL}_2(k), \mathbb{Z}) \to 0$$

or its variant in terms of algebraic K-theory groups

$$0 \to \mu(k) \to K_3^{ind}(k) \to \mathcal{B}(k) \to \bigwedge^2 k^\times \to K_2(k) \to 0.$$

This last sequence, tensored by $\mathbb{Q}$, can be rewritten

$$0 \to K_3^{(2)}(k) \to \mathcal{B}(k) \otimes \mathbb{Q} \to \bigwedge^2 k^\times \to K_2^{(2)}(k) \to 0,$$

where the $K_n^{(i)}(k)$ are the pieces of the Adams decomposition of $K_n(k) \otimes \mathbb{Q}$.

It is likely that considering the extension $\mathcal{B}(k[\varepsilon])$ (see Section 2) of the group $\mathcal{B}(k)$ to the ring $k[\varepsilon]$ of dual numbers, these exact sequences could be generalized as sequences (note that the "weight 1 "part in ε of the first one was investigated in [6])

$$0 \to \mu(k) \to H_3(\mathrm{SL}_2(k[\varepsilon]), \mathbb{Z}) \to \mathcal{B}(k[\varepsilon]) \to \bigwedge^2 k[\varepsilon]^\times \to H_2(\mathrm{SL}_2(k[\varepsilon]), \mathbb{Z}) \to 0,$$

and

$$0 \to K_3^{(2)}(k[\varepsilon]) \to \mathcal{B}(k[\varepsilon]) \otimes \mathbb{Q} \to \bigwedge^2 k[\varepsilon]^\times \to K_2^{(2)}(k[\varepsilon]) \to 0.$$

For algebraically closed fields of characteristic 0, the results of this paper could then be partially recovered, in the following way. Under the action of $k^\times$ in $k[\varepsilon]$: $a + \varepsilon b \mapsto a + \varepsilon t b$, we get weight decompositions of these exact sequences. More precisely, by calculations in cyclic homology [7], the weight decompositions of the K-groups involved reduce to

$$K_3^{(2)}(k[\varepsilon]) = K_3^{(2)}(k) \oplus K_3^{(2)[3]}(k[\varepsilon]),$$
$$K_2^{(2)}(k[\varepsilon]) = K_2(k[\varepsilon]) = K_2^{(2)}(k) \oplus K_2^{(2)[1]}(k[\varepsilon]),$$

where

$$K_3^{(2)[3]}(k[\varepsilon]) \cong k, \quad K_2^{(2)[1]}(k[\varepsilon]) \cong \Omega_k^1.$$

This would imply the following exact sequences, which are in accordance with our results (in particular, the main result below is not a surprise)

$$\begin{array}{rcccl} 0 \to k \to \mathcal{B}(k[\varepsilon])^{[3]} & \to & 0 & & \\ 0 \to \mathcal{B}(k[\varepsilon])^{[2]} & \to & \bigwedge^2 k & \to & 0 \\ 0 \to \mathcal{B}(k[\varepsilon])^{[1]} & \to & k \otimes k^\times & \to & \Omega_k^1 \to 0. \end{array}$$

Note that the two odd weight sequences are analogous to exact sequences appearing in the theory of scissors congruence, connected with the volume and the Dehn invariant (see [9]). The meaning of the one of weight two is not so clear (nevertheless see [19]).

In what follows, we consider the complex

$$\begin{aligned} \mathcal{B}(k[\varepsilon]) &\to \bigwedge^2 k[\varepsilon]^\times \\ u &\mapsto u \wedge (1-u), \end{aligned}$$

for any field. One main result connected to the study of this complex is the following.

THEOREM 1.1. — *Let k be a field. If $\mathrm{Char}(k) \neq 2, 3$, $\mathcal{B}(k[\varepsilon])$ is isomorphic as an abelian group to*

$$\mathcal{B}(k) \bigoplus \beta(k) \bigoplus \bigwedge^2 k \bigoplus k,$$

where the space $\beta(k)$ was introduced in [6] (see Section 8).

In general, there is a filtration $F_0 = \mathcal{B}(k[\varepsilon]) \supset F_1 \supset F_2 \supset F_3$, *such that* $\mathcal{B}(k[\varepsilon]) = \mathcal{B}(k) \oplus F_1$ *and*

1) $F_3 \cong k$, *if* $Char(k) \neq 2, 3$.
2) $F_2/F_3 \cong \bigwedge^2 k$, *if* $Char(k) \neq 2$.
3) $F_1/F_2 \cong \beta(k)$, *without any hypothesis on* k.

For another main result, see Theorem 8.13.

In an appendix we give partial results on the extension to the dual numbers of the trilogarithmic complex of Goncharov.

To conclude this introduction, it is also likely that the fine results of Suslin in [31], which address all fields, can be extended to dual numbers. The results of this paper could then be applied to $K_3^{ind}(k[\varepsilon])$, for general fields.

2. The Bloch-Suslin complex of a commutative ring

For A a commutative ring, let $A^\times$ be the multiplicative group of invertible elements of A. We define $\mathcal{B}(A)$ as the commutative group with generators symbols $\{\alpha\}$, such that $\alpha,\ 1-\alpha \in A^\times$, submitted to the five terms relation

$$\{\alpha\} - \{\alpha'\} + \left\{\frac{\alpha'}{\alpha}\right\} - \left\{\frac{1-\alpha'}{1-\alpha}\right\} + \left\{\frac{\alpha(1-\alpha')}{\alpha'(1-\alpha)}\right\} = 0,$$

where all the elements: $\alpha,\ 1-\alpha,\ \alpha',\ 1-\alpha'$ and $\alpha'-\alpha$ are in $A^\times$. We can then consider the complex

$$\mathcal{B}(A) \xrightarrow{D} \bigwedge^2 A^\times$$

with

$$D(\{\alpha\}) = \alpha \wedge (1-\alpha).$$

This extension of the Bloch-Suslin complex is probably reasonable only for commutative rings with many units.

Let $\mathcal{G}(A)$ (resp. $\mathcal{R}(A)$) be the free $\mathbb{Z}$-module generated by symbols $|\alpha|$, with $\alpha,\ 1-\alpha \in A^\times$ (resp. symbols $|\alpha,\alpha'|$, with $\alpha,\ 1-\alpha,\ \alpha',\ 1-\alpha',\ \alpha'-\alpha \in A^\times$). For further purpose, we note the exact sequence

$$\mathcal{R}(A) \to \mathcal{G}(A) \to \mathcal{B}(A) \to 0$$
$$|\alpha| \mapsto \{\alpha\},$$
$$|\alpha,\alpha'| \mapsto |\alpha'| - |\alpha'| + \left|\frac{\alpha'}{\alpha}\right| - \left|\frac{1-\alpha'}{1-\alpha}\right| + \left|\frac{\alpha(1-\alpha')}{\alpha'(1-\alpha)}\right|.$$