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ON IRREGULAR HOLONOMIC $\mathscr{D}$-MODULES

by

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Abstract. — One proves the existence of a canonical lattice for the meromorphic connections; as a consequence, one obtains the two following results:
First, the fact that such a connection, defined outside a set of codimension 3, can be extended everywhere.
Then, the existence of a global good filtration for the holonomic $\mathscr{D}$-modules.

***Résumé* (Sur les modules holonomes irréguliers).** — On démontre l'existence d'un réseau canonique pour les connexions méromorphes ; on en déduit deux résultats :
D'une part, le fait qu'une telle connexion, définie hors d'un ensemble de codimension 3, se prolonge partout.
D'autre part, l'existence d'une bonne filtration globale pour les $\mathscr{D}$-modules holonomes.

I. Meromorphic connections

1. Introduction. — Let X be a complex analytic manifold of dimension n, and let Z be an analytic hypersurface of X (*i.e.* a closed analytic subset of codimension one at each of its points). We denote by $\mathscr{O}_X$ (resp. Ω^p_X) the sheaf of holomorphic functions on X (resp. the sheaf of holomorphic p-forms on X). We denote also by $\mathscr{O}_X[\star Z]$ the sheaf of meromorphic functions on X with poles on Z: if $f = 0$ is a local equation of Z, one has, with the usual notations $\mathscr{O}_X[\star Z] = \mathscr{O}_X[f^{-1}]$; we put also $\Omega^p_X[\star Z] = \mathscr{O}_X[\star Z] \otimes_{\mathscr{O}_X} \Omega^p_X$. Sometimes, we omit "$X$" and we write $\mathscr{O}$, $\mathscr{O}[\star Z]$, Ω^p, etc.

It is well know that $\mathscr{O}$ has noetherian fibers, and that it is coherent (*i.e.* the kernel of a map $\mathscr{O}^q \to \mathscr{O}^p$ is locally of finite type); from this follows at once that $\mathscr{O}[\star Z]$ has the same properties. Then, one defines a $\mathscr{O}[\star Z]$ coherent module, in the usual way, as being locally the cokernel of a morphism of $\mathscr{O}[\star Z]$ modules, say $\mathscr{O}[\star Z]^q \xrightarrow{u} \mathscr{O}[\star Z]^p$.

Let E be a coherent $\mathscr{O}[\star Z]$-module. By definition a lattice of E is a coherent sub-$\mathscr{O}$-module $L \subset E$ such that $E = \mathscr{O}[\star Z]L$. Locally, E admits always lattices (take a

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presentation $u : \mathscr{O}[\star Z]^q \to \mathscr{O}[\star Z]^p \to E \to 0$, and multiply by f^r, $r \gg 0$, to remove the poles...). But E does not admit always global lattices, even if X is compact : see counterexamples below, in subsection 6.

Remark. — Suppose that X is a projective manifold, *i.e.* a closed analytic submanifold of $\mathbf{P}^n(\mathbf{C})$. Then, by a classical theorem of Chow, X "is algebraic", *i.e.* there exists a projective algebraic manifold $\widetilde{X}$ such that $X = \widetilde{X}^{\mathrm{an}}$, the analytic manifold associated to $\widetilde{X}$.

Now, let Z be an hypersurface of X, which is also "algebraic" in the same sense, and let E be a coherent $\mathscr{O}[\star Z]$-module; then, the following assertions are equivalent:

i) E admits a lattice L

ii) E is "algebraic", *e.g.* there exists a $\mathscr{O}_{\widetilde{X}}[\star \widetilde{Z}]$-module $\widetilde{E}$ such that $E = \widetilde{E}^{\mathrm{an}}$ ($= \widetilde{E} \otimes_{\mathscr{O}_{\widetilde{X}}} \mathscr{O}_X$)

i) $\Rightarrow$ ii) follows from "GAGA", which asserts that the coherent $\mathscr{O}_X$-modules "are algebraic"

ii) $\Rightarrow$ i) follows from a standard result of algebraic geometry with asserts that quasi-coherent sheaves on algebraic varieties which some mild finiteness assumptions (in particular, projective algebraic varieties) are inductive limits of coherent sheaves.

Now, we come back to the general case.

Definition 1.1. — Let E be a coherent $\mathscr{O}[\star Z]$-module. A connection on E is defined, in the usual way, as an operator $\nabla : E \to E \otimes_{\mathscr{O}} \Omega^1$ verifying the following properties

i) ∇ is $\mathbf{C}$-linear

ii) For $\phi \in \mathscr{O}$, $e \in E$ one has $\nabla(\phi e) = e \otimes d\phi + \phi \nabla e$

For such a ∇, one defines as usual its extension (denoted also ∇): $E \otimes_{\mathscr{O}} \Omega^p \to E \otimes_{\mathscr{O}} \Omega^{p+1}$. One says that ∇ is flat if $\nabla^2 : E \to E \otimes_{\mathscr{O}} \Omega^2$ vanishes; in that case $\nabla^2 : E \otimes_{\mathscr{O}} \Omega^p \to E \otimes_{\mathscr{O}} \Omega^{p+2}$ vanishes also for all p (proofs as usual in differential geometry).

Proposition 1.2. — *Let E be a coherent $\mathscr{O}[\star Z]$-module, with a connection ∇ (non necessary flat). Then, E is locally stably free, i.e. , for every point $x \in X$, there exists p such that $E_x \oplus \mathscr{O}_x[\star Z]^p$ is free*

The proof is similar to the proof of proposition 2.2 below (and in fact simpler!).

The main result of this chapter is the following : a coherent $\mathscr{O}[\star Z]$-module provided with a flat connection admits a global lattice and actually admits a canonical lattice; see the precise statement in section 4. To prove this result, we need the use of formal completions, which we will study now.

2. Formal completions. — Let X, Z, $\mathscr{O}_X, \ldots,$ be as before. We define, as usual, the formal completion $\widehat{\mathscr{O}}_{X|Z}$ (or $\widehat{\mathscr{O}}$, if there is no ambiguity) as the sheaf on Z associated to the presheaf $U \to \varprojlim \Gamma(U, \mathscr{O}/f^k\mathscr{O})$ with f a local equation of Z. It is obvious that $\widehat{\mathscr{O}}_{X|Z,a}$ is contained in the formal completion of $\mathscr{O}_a$ with respect to the powers of the maximal ideal $\mathscr{M}_a$; in particular, $\widehat{\mathscr{O}}_{X|Z,a}$ is an integral domain. Furthermore, it is noetherian and faithfully flat over $\mathscr{O}_a$; also $\widehat{\mathscr{O}}_{X|Z}$ is coherent.

I shall not prove these properties here, although I have no explicit reference. To prove that $\widehat{\mathscr{O}}_{X|Z,a}$ is noetherian and that $\widehat{\mathscr{O}}_{X|Z}$ is coherent one can for instance argue as in [**L-M**], where a more delicate case of formal completions is treated; the main ingredients are the "theorem of privileged neighborhoods" and the theorem of Frisch asserting that the ring of holomorphic functions on a closed polycylinder is noetherian. Then, the faithful flatness of $\widehat{\mathscr{O}}_a$ onto $\mathscr{O}_a$ follows from the fact that they have same completion (for the topology defined by the powers of the maximal ideal).

One defines $\widehat{\mathscr{O}}[*Z]$, $\widehat{\Omega}^p$, etc. as before. If F is a coherent $\widehat{\mathscr{O}}[*Z]$-module, one defines also a lattice of F as a coherent sub-$\widehat{\mathscr{O}}$-module L such that $F = \widehat{\mathscr{O}}[*Z]L$.

Let E be a coherent $\mathscr{O}[*Z]$-module, and put $\widehat{E} = E \otimes_{\mathscr{O}} \widehat{\mathscr{O}}$; if L is a lattice of E, $\widehat{L} = L \otimes_{\mathscr{O}} \widehat{\mathscr{O}}$ is a lattice of $\widehat{E}$. Note also that the natural map $E \to \widehat{E}$ is injective (use the exact sequence $0 \to \mathscr{O}_a \to \widehat{\mathscr{O}}_a \to \widehat{\mathscr{O}}_a/\mathscr{O}_a \to 0$ and the fact that $\operatorname{Tor}_1^{\mathscr{O}_a}(E, \widehat{\mathscr{O}}_a/\mathscr{O}_a) = 0$).

Proposition 2.1. — *The mapping $L \to \widehat{L}$ is a bijection "lattices of E" $\simeq$ "lattices of $\widehat{E}$". The inverse is the mapping $M \to M \cap E$ (extended by E outside Z).*

One proves this result as follows: as the result is local, one can suppose that one has already a lattice L' of E, and one equation $f = 0$ of Z. Then, locally, if L is a lattice of E, one has, for some q : $f^qL' \subset L$. Similarly, if M is a lattice of $\widehat{E}$, one has locally $f^q\widehat{L'} \subset M$. But the lattices of E (resp. $\widehat{E}$) which contain f^qL' (resp $f^q\widehat{L'}$) are in one to one correspondence with the coherent $\mathscr{O}$-sub modules of E/f^qL' (resp. with the coherent $\widehat{\mathscr{O}}$-sub modules of $\widehat{E}/f^q\widehat{L}')$; then, the result follows from the equality $E/f^qL' \simeq \widehat{E}/f^q\widehat{L}'$.

Let now F be a coherent $\widehat{\mathscr{O}}[*Z]$-module. One defines a connection ∇ on F as before, in the case of $\mathscr{O}[*Z]$-modules.

Proposition 2.2. — *Let F be a coherent $\widehat{\mathscr{O}}[*Z]$-module provided with a connection (not necessarily flat). Then, F is locally stably free.*

To prove this proposition, we need a lemma.

Lemma 2.3. — *For $i \geqslant 1$, one has $\operatorname{Ext}^i_{\widehat{\mathscr{O}}[*Z]_a}(F_a, \widehat{\mathscr{O}}[*Z]_a) = 0$ (a, a point of Z)*

Denote by P one of these Ext^i; it is finite over $\widehat{\mathscr{O}}[*Z]_a$ and is a torsion module (since it is annihilated by extension of $\widehat{\mathscr{O}}[*Z]_a$ to its fraction field). On the other hand, it is naturally provided with a connection: to prove this, we take an injective resolution $I^{\cdot}$ of $\widehat{\mathscr{O}}[*Z]_a$ over $\widehat{\mathscr{D}}[*Z]_a =$ the ring of differential operators with coefficients in $\widehat{\mathscr{O}}[*Z]_a$;

this resolution is also injective over $\widehat{\mathscr{O}}[*Z]_a$ (exercise: use the fact that $\widehat{\mathscr{D}}[*Z]_a$ in flat over $\widehat{\mathscr{O}}[*Z]_a$); then, one consider the obvious connection on $\mathrm{Hom}_{\widehat{\mathscr{O}}[*Z]_a}(F_a, I^k)$ and the cohomology groups of the corresponding complex.

Take now $g \in \mathrm{Ann}\, P$, and take $p \in P$; in local coordinates, one has $(\partial_i g)p + g(\nabla_{\partial_i} p) = 0$ ($\partial_i = \partial/\partial x_i$); therefore one has $(\partial_i g)p = 0$. Therefore $\mathrm{Ann}\, P$ is stable by the derivations ∂_i. As $\mathrm{Ann}\, P \neq 0$, it implies that one has $\mathrm{Ann}\, P = \widehat{\mathscr{O}}[*Z]_a$. (Exercise: choose a $g \in \mathrm{Ann}\, P$; multiplying it by f^p, $p \gg 0$, we can suppose that g has no pole; then, develop it in power series at a, and find a differential operator $b(x, \partial)$ such that $b(x, \partial)g$ is invertible in $\widehat{\mathscr{O}}_a$). Therefore one has $P = 0$ and the lemma is proved.

Now, the proof of the proposition follows a standard line. First, note the following facts.

i) The theorem of syzygies is true for $\widehat{\mathscr{O}}_a$, *i.e.* a finite module E over $\widehat{\mathscr{O}}_a$ has a free resolution of finite length (actually of length $\leqslant n = \dim X$). As $\widehat{\mathscr{O}}_a$ is local and noetherian, a standard argument shows that it suffices to prove the result for $E = \mathbf{C} = \widehat{\mathscr{O}}_a/\widehat{\mathscr{M}_a}$ ($\widehat{\mathscr{M}_a}$, the maximal ideal of $\widehat{\mathscr{O}}_a$); but this follows at once from the same result for $\mathscr{O}_a$, and the fact that $\widehat{\mathscr{O}}_a$ is flat over $\mathscr{O}_a$.

ii) From this, follows that the theorem of syzygies is also true for $\widehat{\mathscr{O}}[*Z]_a$; in fact, take E finite over $\widehat{\mathscr{O}}[*Z]_a$, and choose a lattice $L \subset E$, *i.e.* a finite $\widehat{\mathscr{O}}_a$ submodule such that $\widehat{\mathscr{O}}[*Z]_a L = E$; the natural mapping $L \otimes_{\widehat{\mathscr{O}}_a} \widehat{\mathscr{O}}[*Z]_a \to E$ is bijective [the surjectivity is obvious; to prove the injectivity, note *e.g.* that the map $L \otimes_{\widehat{\mathscr{O}}_a} \widehat{\mathscr{O}}[*Z]_a \to E \otimes_{\widehat{\mathscr{O}}_a} \widehat{\mathscr{O}}[*Z]_a$ is injective since $\widehat{\mathscr{O}}[*Z]_a$ is flat over $\widehat{\mathscr{O}}_a$; on the other hand, the second term is equal to E: we leave the verification as an exercise]. Now, take a free resolution $\Phi^{\cdot}$ of L over $\widehat{\mathscr{O}}_a$; then $\Phi^{\cdot} \otimes_{\widehat{\mathscr{O}}_a} \widehat{\mathscr{O}}[*Z]_a$ is a free resolution of E over $\widehat{\mathscr{O}}[*Z]_a$; and if $\Phi^{\cdot}$ has finite length, the last one has also finite length, which proves ii).

Now, the proof of the proposition is done in two steps; take F as in the proposition 2.2, and take $a \in Z$.

i) F_a is projective; it is sufficient to prove the following result: if E is finite over $\widehat{\mathscr{O}}[*Z]_a$, then one has, for $i \geqslant 1$: $\mathrm{Ext}^i_{\widehat{\mathscr{O}}[*Z]_a}(F_a, E) = 0$. One proves this result by induction on the length of a free resolution of E. If *e.g.* E admits a resolution of length ℓ, one has an exact sequence $0 \to E' \to \widehat{\mathscr{O}}[*Z]_a^p \to E \to 0$, where E' has a free resolution of length $\ell - 1$; then the exact sequence of "Ext" imply that $\mathrm{Ext}^i(F_a, E) = \mathrm{Ext}^{i+1}(F_a, E')$ $(i \geqslant 1)$, and the result follows.

ii) Any projective module G of finite type over $\widehat{\mathscr{O}}[*Z]_a$ is stably free. This is proved by induction on the length of a free resolution of G: if G has a free resolution of length ℓ, one has an exact sequence $O \to G' \to \widehat{\mathscr{O}}[*Z]_a^p \to G \to 0$, where G' has a free resolution of length $\ell - 1$; G being projective, the exact sequence splits, and G' is also projective. Then, the result follows from the induction hypothesis.

Remark. — I do not know if a stably free module of finite type over $\mathscr{O}[*Z]_a$ or $\widehat{\mathscr{O}}_{X|Z}[*Z]_a$ is actually free (of course, on $\mathscr{O}_a$ or $\mathscr{O}_{X|Z,a}$, this is true since these are local rings). If instead of $\mathscr{O}[*Z]_a$, we have a ring of polynomials $\mathbf{C}[x_1,\ldots,x_n]$, then the similar statement is true according to a celebrated theorem of Quillen-Suslin. But I do not know if their methods can be extended to the cases considered here.

3. Extension of coherent sheaves. — Let X be an analytic manifold of dimension n, and let S be a closed analytic subset of X, of codimension $\geqslant 2$; we denote by i the injection $X - S \to X$. As before, $\mathscr{O}_X$ denotes the sheaf of holomorphic functions on X. Recall first the following result.

***Proposition 3.1* (“Hartogs property”)**. — *The natural morphism $\mathscr{O}_X \to i_*\mathscr{O}_{X-S}$ is an isomorphism.*

In other words, if a is a point of S and U an open neighborhood of a in X, then a holomorphic function on $U - S$ extends in a unique way to a holomorphic function on U. When S is smooth, this follows from a classical argument of Hartogs; in general, the result follows by using a stratification of S by smooth subvarieties, and by an argument of decreasing induction on the dimension of the strata.

Given an $\mathscr{O}_X$-coherent sheaf F, we call $F^\vee = \mathit{Hom}_{\mathscr{O}_X}(F,\mathscr{O}_X)$ the dual of F; we say that F is reflexive if the natural mapping $F \to F^{\vee\vee}$ is bijective. The following proposition is a simple particular case of a result of Serre [**Se**].

Proposition 3.2. — *Let F be a coherent $\mathscr{O}_{X-S}$-module, which is reflexive. Then, the following properties are equivalent:*

i) F admits a coherent extension to X (in that case, we say that “F is extendable”)

ii) i_*F is coherent.

The assertion “ii) implies i)” is obvious. Conversely, suppose that F is extendable; then $G = \mathit{Hom}_{\mathscr{O}_X}(F,\mathscr{O}_X)$ admits also an extension, say $\overline{G}$; I claim that $\overline{G}^\vee = i_*F$; actually, one has

$$\begin{aligned}\overline{G}^\vee &= \mathit{Hom}_{\mathscr{O}_X}(\overline{G},\mathscr{O}_X)\\ &= \mathit{Hom}_{\mathscr{O}_X}(\overline{G}, i_*\mathscr{O}_{X-S}) \quad &\text{(“Hartogs property”)}\\ &= i_*\,\mathit{Hom}_{\mathscr{O}_{X-S}}(i^*\overline{G},\mathscr{O}_{X-S}) \quad &\text{(adjunction)}\\ &= i_*G^\vee\\ &= i_*F\end{aligned}$$

[For the adjunction formula which is purely sheaf-theoretic, we refer to the standard literature on sheaves]

In section 5, we will see deeper results on extension of sheaves; note that one interest of the property is the following fact : under the condition of prop. 3.2, the fact for a sheaf to be extendable is a local property.

We will now consider similar results for formal completions; as in section 1, let $Z \subset X$ be a closed hypersurface, let $S \subset Z$ be a closed analytic subset of codimension $\geqslant 2$ (with respect to X). We denote j the injection $Z - S \to Z$. The analogue of proposition 3.1 is here the following statement.