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Astérisque, tome 261 (2000), p. 405-443

<http://www.numdam.org/item?id=AST_2000__261__405_0>

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RATIONAL PARAMETER RAYS OF THE MANDELBROT SET

by

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Abstract. — We give a new proof that all external rays of the Mandelbrot set at rational angles land, and of the relation between the external angle of such a ray and the dynamics at the landing point. Our proof is different from the original one, given by Douady and Hubbard and refined by P. Lavaurs, in several ways: it replaces analytic arguments by combinatorial ones; it does not use complex analytic dependence of the polynomials with respect to parameters and can thus be made to apply for non-complex analytic parameter spaces; this proof is also technically simpler. Finally, we derive several corollaries about hyperbolic components of the Mandelbrot set.

Along the way, we introduce partitions of dynamical and parameter planes which are of independent interest, and we interpret the Mandelbrot set as a symbolic parameter space of kneading sequences and internal addresses.

1. Introduction

Quadratic polynomials, when iterated, exhibit amazingly rich dynamics. Up to affine conjugation, these polynomials can be parametrized uniquely by a single complex variable. The Mandelbrot set serves to organize the space of (conjugacy classes of) quadratic polynomials. It can be understood as a "table of contents" to the dynamical possibilities and has a most beautiful structure. Much of this structure has been discovered and explained in the groundbreaking work of Douady and Hubbard [**DH1**], and a deeper understanding of the fine structure of the Mandelbrot set is a very active area of research. The importance of the Mandelbrot set is due to the fact that it is the simplest non-trivial parameter space of analytic families of iterated holomorphic maps, and because of its universality as explained by Douady and Hubbard [**DH2**]: the typical local configuration in one-dimensional complex parameter spaces is the Mandelbrot set (see also [**McM**]).

1991 ***Mathematics Subject Classification.*** — 30C20, 30D05, 30D40.
Key words and phrases. — Mandelbrot set, kneading sequence, internal address, external ray, parameter ray, hyperbolic component.

Unfortunately, most of the beautiful results of Douady and Hubbard on the structure of the Mandelbrot set are written only in preliminary form in the preprints **[DH1]**. The purpose of this article is to provide concise proofs of several of their theorems. Our proofs are quite different from the original ones in several respects: while Douady and Hubbard used elaborate perturbation arguments for many basic results, we introduce partitions of dynamical and parameter planes, describe them by symbolic dynamics, and reduce many of the questions to a combinatorial level. We

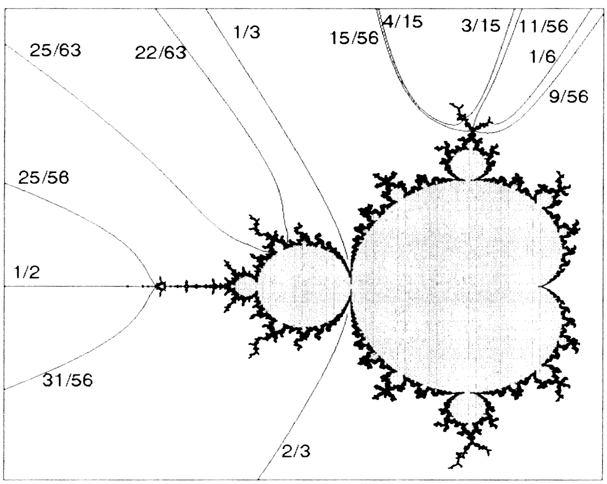

FIGURE 1. *The Mandelbrot set and several of its parameter rays which are mentioned in the text. Picture courtesy of Jack Milnor.*

feel that our proofs are technically significantly simpler than those of Douady and Hubbard. An important difference for certain applications is that our proof does not use complex analytic dependence of the maps with respect to the parameter and is therefore applicable in certain wider circumstances: the initial motivation for this research was a project with Nakane (see **[NS]** and the references therein) to understand the parameter space of antiholomorphic quadratic polynomials, which depends only real-analytically on the parameter. Of course, the "standard proof" using Fatou coordinates and Ecalle cylinders, as developed by Douady and Hubbard and elaborated by Lavaurs [**La2**], is a most powerful tool giving interesting insights; it has had many important applications. Our goal is to present an alternative approach in order to enlarge the toolbox for applications in different situations.

The fundamental result we want to describe in this article is the following theorem about landing properties of external rays of the Mandelbrot set, a theorem due to Douady and Hubbard; for background and terminology, see the next section.

Theorem 1.1 **(The Structure Theorem of $\mathcal{M}$).** — *Parameter rays of the Mandelbrot set at rational angles have the following landing properties:*

(1) *Every parameter ray at a periodic angle ϑ lands at a parabolic parameter such that, in its dynamic plane, the dynamic ray at angle ϑ lands at the parabolic orbit and is one of its two characteristic rays.*
(2) *Every parabolic parameter c is the landing point of exactly two parameter rays at periodic angles. These angles are the characteristic angles of the parabolic orbit in the dynamic plane of c.*
(3) *Every parameter ray at a preperiodic angle ϑ lands at a Misiurewicz point such that, in its dynamic plane, the dynamic ray at angle ϑ lands at the critical value.*
(4) *Every Misiurewicz point c is the landing point of a finite non-zero number of parameter rays at preperiodic angles. These angles are exactly the external angles of the dynamic rays which land at the critical value in the dynamic plane of c.*

(The parameter $c = 1/4$ is the landing point of a single parameter ray, but this ray corresponds to external angles 0 and 1; we count this ray twice in order to avoid having to state exceptions.)

The organization of this article is as follows: in Section 2, we describe necessary terminology from complex dynamics and give a few fundamental lemmas. Section 3 contains a proof of the periodic part of the theorem, and along the way it shows how to interpret the Mandelbrot set as a parameter space of kneading sequences. The preperiodic part of the theorem is then proved in Section 4, using properties of kneading sequences. In the final Section 5, we derive fundamental properties of hyperbolic components of the Mandelbrot set. Most of the results and proofs in this paper work also for "Multibrot sets": these are the connectedness loci of the polynomials $z^d + c$ for $d \geq 2$.

This article is an elaborated version of Chapter 2 of my Ph.D. thesis [**S1**] at Cornell University, written under the supervision of John Hubbard and submitted in the summer of 1994. It is part of a mathematical ping-pong with John Milnor: it builds at important places on the paper [**GM**]; recently Milnor has written a most beautiful new paper **[M2]** investigating external rays of the Mandelbrot set from the point of view of "orbit portraits", i.e., landing patterns of periodic dynamic rays. I have not tried to hide how much both I and this paper have profited from many discussions with him, as will become apparent at many places. This paper, as well as Milnor's new one, uses certain global counting arguments to provide estimates, but in different directions. It is a current project **[ES]** to combine both approaches to provide a new, more conceptual proof without global counting. Proofs in a similar spirit of further fundamental properties of the Mandelbrot set can be found in **[S2]**.

Acknowledgements. — It is a pleasure to thank many people who have contributed to this paper in a variety of ways. I am most grateful to John Hubbard for so much help and friendship over the years. To John Milnor, I am deeply indebted for the ping-pong mentioned above, as well as for many discussions and a lot of encouragement along the way. Tan Lei contributed very helpful corrections and suggestions most constructively. Many more people have shared their ideas and understanding with me and will recognize their contributions; they include Bodil Branner, Adrien Douady, Karsten Keller, Eike Lau, Jiaqi Luo, Misha Lyubich, Shizuo Nakane, Chris Penrose, Carsten Petersen, Johannes Riedl, Mitsu Shishikura and others. I am also grateful to the Institute for Mathematical Sciences in Stony Brook for its inspiring environment and support. Finally, special thanks go to Katrin Wehrheim for a most helpful suggestion.

2. Complex Dynamics

In this section, we briefly recall some results and notation from complex dynamics which will be needed in the sequel. For details, the notes **[M1]** by Milnor are recommended and, of course, the work **[DH1]** by Douady and Hubbard which is the source of most of the results mentioned below.

By affine conjugation, quadratic polynomials can be written uniquely in the normal form $p_c : z \mapsto z^2 + c$ for some complex parameter c. For any such polynomial, the filled-in Julia set is defined as the set of points z with bounded orbits under iteration. The Julia set is the boundary of the filled-in Julia set. It is also the set of points which do not have a neighborhood in which the sequence of iterates is normal (in the sense of Montel). Julia set and filled-in Julia set are connected if and only if the only critical point 0 has bounded orbit; otherwise, these sets coincide and are a Cantor set. The Mandelbrot set $\mathcal{M}$ is the *quadratic connected locus*: the set of parameters c for which the Julia set is connected. Julia sets and filled-in Julia sets, as well as the Mandelbrot set, are compact subsets of the complex plane. The Mandelbrot set is known to be connected and full (i.e. its complement is connected).

Douady and Hubbard have shown that Julia sets and the Mandelbrot set can profitably be studied using external rays: for a compact connected and full set $K \subset \mathbb{C}$,