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THE LOCAL LANGLANDS CORRESPONDENCE: NOTES OF (HALF) A COURSE AT THE IHP SPRING 2000

by

Michael Harris

Abstract. — The article provides a reasonably self-contained account of the main results of the author's book with Richard Taylor, containing a description of the Galois representations obtained in the cohomology of certain Shimura varieties attached to unitary groups, and obtaining the local Langlands conjecture for $GL(n)$ of p-adic fields as a consequence. The main steps in the proof of the local Langlands conjecture are presented, and in some cases simplified. The bulk of the paper concerns the geometry of the Shimura variety at places of bad reduction, where good local models are nevertheless available, and the description of points in the special fiber in the manner of Langlands and Kottwitz. The article concludes with a section describing possible extensions of these results to other Shimura varieties, and an account of some of the work of Laurent Fargues along those lines.

***Résumé* (La correspondance de Langlands locale).** — L'article contient une description assez complète des principaux résultats du livre de l'auteur avec Richard Taylor, qui décrit les représentations galoisiennes réalisées dans la cohomologie de certaines variétés de Shimura associées aux groupes unitaires, et qui obtient la conjecture locale de Langlands pour $GL(n)$ d'un corps p-adique comme conséquence. Les principales étapes de la démonstration de la conjecture locale de Langlands y sont présentées, parfois simplifiées. Le gros de l'article concerne la géométrie de la variété de Shimura aux places de mauvaise réduction, où l'on dispose néanmoins de bons modèles locaux, et la description des points dans la fibre spéciale à la manière de Langlands et Kottwitz. La dernière section de l'article décrit les extensions éventuelles de ces résultats aux variétés de Shimura plus générales, ainsi qu'un compte rendu des travaux de Laurent Fargues sur ces questions.

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Introduction

The present notes cover 50% of the material presented in a course given jointly with Guy Henniart during the special semester "Formes Automorphes", held at the Institut Henri Poincaré in Paris between February and June 2000, as well as a little more material I didn't have time to present. The purpose of the course was to explain two proofs of the local Langlands conjecture for p-adic fields, due respectively to Richard Taylor and myself [**HT**], and to Henniart [**He5**]. My lectures were naturally concerned with [**HT**], the main burden of which is to construct a candidate for a local Langlands correspondence, and to prove that this putative correspondence is (nearly) compatible with the global correspondence realized on the cohomology of certain specific Shimura varieties. The techniques applied derive mainly from arithmetic algebraic geometry: we study the bad reduction of the Shimura varieties in question by interpreting them locally/infinitesimally as formal deformation spaces for p-divisible groups with additional structure of a kind already studied by Drinfel'd. This yields a stratification of the special fiber, with particularly nice properties, in terms of p-rank of the universal p-divisible group.The cohomology of the Shimura varieties is then calculated by means of vanishing cycles on the bad special fiber. Thanks to Berkovich's work on étale cohomology of (rigid) analytic spaces, the vanishing cycles can be computed infinitesimally, which permits determination of their stalks in terms of certain universal representation spaces. An extension, to our situation of bad reduction, of the trace formula techniques perfected by Langlands and Kottwitz for calculating zeta functions of Shimura varieties at places of good reduction, provides the necessary compatibility of local and global correspondences.

My goal in the course was to present a self-contained account of the main results of [**HT**]. In so doing, I chose to sacrifice the description of the global structure of the strata in the special fiber, and of the vanishing cycles sheaves on the strata, in the first place because this would have been impossible in the eight three-hour sessions available, but also because no such description seemed likely to be available for other Shimura varieties.(1) My presentation therefore diverged from that of [**HT**], in that I studied the vanishing cycles by means of formal completions along points in the special fiber, following the approach of Rapoport and Zink in [**RZ**], rather than along the strata. This was nearly successful: the geometric material was covered in detail, but I ran out of time and was unable to do justice to the detailed comparison of trace formulas. This was just as well, because I did not find a satisfactory approach to the latter – an approach likely to extend to other groups – until long after the semester had ended and all the visitors had gone home.(2) This is the approach presented in the present notes.

(1) In the meantime, Elena Mantovan's Harvard Ph.D. thesis [**Ma**] has revealed this expectation to be unduly pessimistic.

(2) To be honest, talking to the visitors was much more interesting than perfecting the final stages of the argument.

We introduce the notation that will be used throughout these notes. Let p be a rational prime number. For any finite extension K of $\mathbb{Q}_p$ and any positive integer n, we let $\mathcal{A}(n, K)$ denote the set of equivalence classes of irreducible admissible representations of $GL(n, K)$, $\mathcal{A}_0(n, K)$ the subset of supercuspidal representations. Let $\mathcal{G}(n, K)$ denote the set of equivalence classes of n-dimensional complex representations of the Weil-Deligne group WD_K on which Frobenius acts semisimply, $\mathcal{G}_0(n, K)$ the subset of irreducible representations. We will frequently write G_n for $GL(n)$.

A *local Langlands correspondence* for p-adic fields is the following collection of data:

(0.1) For every p-adic field and integer $n \geqslant 1$, a bijection $\pi \to \sigma(\pi)$ between $\mathcal{A}(n, K)$ and $\mathcal{G}(n, K)$ that identifies $\mathcal{A}_0(n, K)$ with $\mathcal{G}_0(n, K)$.

(0.2) Let χ be a character of $K^\times$, which we identify with a character of WD_K via the reciprocity isomorphism of local class field theory. Then $\sigma(\pi \otimes \chi \circ \det) = \sigma(\pi) \otimes \chi$. In particular, when n = 1, the bijection is given by local class field theory.

(0.3) If $\pi \in \mathcal{A}(n, K)$ with central character $\xi_\pi \in \mathcal{A}(1, K)$, then $\xi_\pi = \det(\sigma(\pi))$.

(0.4) $\sigma(\pi^\vee) = \sigma(\pi)\vee$, where $^\vee$ denotes contragredient.

(0.5) Let $\alpha : K \to K_1$ be an isomorphism of local fields. Then α induces bijections $\mathcal{A}(n, K) \to \mathcal{A}(n, K_1)$ and $\mathcal{G}(n, K) \to \mathcal{G}(n, K_1)$ for all n, and we have $\sigma(\alpha(\pi)) = \alpha(\sigma(\pi))$. In particular, if K is a Galois extension of a subfield K_0, then the bijection σ respects the $\mathrm{Gal}(K/K_0)$-actions on both sides.

(0.6) Let K'/K denote a cyclic extension of prime degree d. Let $BC : \mathcal{A}(n, K) \to \mathcal{A}(n, K')$ and $AI : \mathcal{A}(n, K') \to \mathcal{A}(nd, K')$ denote the local base change and automorphic induction maps [**AC, HH**]. Let $\pi \in \mathcal{A}(n, K)$, $\pi' \in \mathcal{A}(n, K')$. Then

$$\sigma(BC(\pi)) = \sigma(\pi)|_{WD_{K'}} \tag{0.6.1}$$

$$\sigma(AI(\pi')) = \mathrm{Ind}_{K'/K}\, \sigma(\pi'), \tag{0.6.2}$$

where $\mathrm{Ind}_{K'/K}$ denotes induction from $WD_{K'}$ to WD_K.

Let n and m be positive integers, $\pi \in \mathcal{A}(n, K)$, $\pi' \in \mathcal{A}(m, K)$. Then

(0.7) $L(s, \pi \otimes \pi') = L(s, \sigma(\pi) \otimes \sigma(\pi'))$.

(0.8) For any additive character ψ of K, $\varepsilon(s, \pi \otimes \pi', \psi) = \varepsilon(s, \sigma(\pi) \otimes \sigma(\pi'), \psi)$.

Here the terms on the left of (0.7) and (0.8) are as in [**JPSS, Sh**] and are compatible with the global functional equation for Rankin-Selberg L-functions. The right-hand terms are given by Artin and Weil (for (0.7)) and Langlands and Deligne (for (0.8)) and are compatible with the functional equation of L-functions of representations of the global Weil group.In particular both sides have Artin conductors and (0.8) implies that $a(\sigma(\pi)) = a(\pi)$.

The local Langlands conjecture, established in [**HT**] and in [**He5**], is the assertion that a local Langlands correspondence exists. The existence of some family of bijections $\mathcal{A}(n, K) \leftrightarrow \mathcal{G}(n, K)$, identifying $\mathcal{A}_0(n, K)$ with $\mathcal{G}_0(n, K)$, preserving conductors and satisfying weakened versions of properties (0.2)-(0.5), had been proved by Henniart a number of years before [**He2**]. Henniart's main tools are a counting argument for local fields of positive characteristic, based on Laumon's theory of the ℓ-adic Fourier transform (the subsets of $\mathcal{A}_0(n, K)$ and $\mathcal{G}_0(n, K)$ with fixed conductor are finite) and an "approximation" of local fields of characteristic zero by local fields of positive characteristic. The properties established in [**He2**] do not suffice to characterize the correspondence uniquely. However, another theorem of Henniart ([**He4**]; *cf.* (A.2.5), below) guarantees that properties (0.1)-(0.8) do suffice to determine a unique correspondence.(3) Nevertheless, the "numerical local Langlands correspondence" of [**He2**] is a necessary ingredient of all proofs to date of the local Langlands correspondence in mixed characteristic. In the present notes, it is invoked in (5.3).(4)

The notes are divided into eight more or less fictitious lectures, following my original plan which proved too ambitious; even the first seven lectures did not fit in the time allotted. The first lecture covers the arguments common to [**HT**] and [**He5**]: the construction of special families of cohomological automorphic representations of $GL(n)$ of CM fields, corresponding to certain cases of non-Galois automorphic induction of Hecke characters. These arguments are mostly taken from [**H2**], which uses these special automorphic representations to reduce the local Langlands conjecture – more precisely, property (0.8), the others being established by geometric means – to the local/global compatibility, asserted as Main Theorem 1.3.6.

The next three lectures present an attenuated version of the geometric part of [**HT**]. The main object of these notes is the Shimura variety attached to the unitary (similitude) group G of a division algebra of dimension n^2 over a CM field F, with involution of the second kind fixing the real subfield F^+ of F. As complex analytic varieties, they are compact quotients of the unit ball of dimension $n-1$. Lecture 2 introduces these Shimura varieties as moduli spaces of abelian varieties with PEL type. Their regular integral models in ramified level, over a p-adic place w of F split over F^+, are defined by means of Drinfel'd bases. The main properties of the latter are recalled in Lecture 3, which also carries out the thankless task of explaining how Hecke operators act on Drinfel'd bases. The stratification by p-rank of the special fiber at a split place is defined in Lecture 4: it is shown that there is one stratum, a union of locally closed smooth subvarieties, in each dimension $h = 0, 1, \dots, n-1$. Infinitesimal uniformization, as in [**RZ**], is then combined with the results of Berkovich to show that the stalks of the vanishing cycles sheaves are constant along strata, and are isomorphic on the h-dimensional stratum to a standard space Φ_{n-h} with canonical action of $GL(n-h, F_w) \times J_{n-h}$, where J_{n-h} is a specific anisotropic inner form of $GL(n-h)$ over F_w.

(3) Of course the local Langlands conjecture is originally due to Langlands! The form presented here became standard after it was understood that conditions (0.7) and (0.8) for $m = 1$ do not suffice to characterize the correspondence.

(4) In his IHP lectures, Henniart replaced the counting argument in positive characteristic by a reference to Lafforgue's theorem which establishes the global Langlands correspondence for function fields, with the local Langlands correspondence in positive characteristic as a corollary. The original proof [**LRS**] of the local Langlands correspondence in positive characteristic used [**He2**].