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ON ISOPERIMETRIC INEQUALITY IN ARAKELOV GEOMETRY

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ABSTRACT. — We establish an isoperimetric inequality in an integral form and deduce a relative version of Brunn-Minkowski inequality in the Arakelov geometry setting.

RÉSUMÉ (*Sur l'inégalité isopérimétrique en géométrie d'Arakelov*). — On établit une inégalité isopérimétrique sous une forme d'intégration dans le cadre de géométrie d'Arakelov et en déduit une version relative de l'inégalité de Brunn-Minkowski dans le même cadre.

1. Introduction

The isoperimetric inequality in Euclidean geometry asserts that, for any convex body Δ in $\mathbb{R}^d$, one has

$$\operatorname{vol}(\partial\Delta)^d \geq d^d \operatorname{vol}(B) \operatorname{vol}(\Delta)^{d-1}, \tag{1}$$

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where B denotes the closed unit ball in $\mathbb{R}^d$. From the point of view of convex geometry, the isoperimetric inequality can be deduced from the Brunn-Minkowski inequality: for two Borel subsets A_1 and A_2 in $\mathbb{R}^d$, one has

$$\mathrm{vol}(A_0 + A_1)^{1/d} \geq \mathrm{vol}(A_0)^{1/d} + \mathrm{vol}(A_1)^{1/d}, \tag{2}$$

where

$$A_0 + A_1 := \{x + y | x \in A_0, y \in A_1\}$$

is the Minkowski sum of A_0 and A_1. The proof consists of taking $A_0 = \Delta$ and $A_1 = \varepsilon B$ in (2) with $\varepsilon > 0$ and letting ε tend to 0. We refer readers to [35] for a presentation on the history of the isoperimetric inequality and to page 1190 of *loc. cit.* for more details on how to deduce (1) from (2). The same method actually leads to a lower bound for the mixed volume of convex bodies:

$$\mathrm{vol}_{d-1,1}(\Delta_0, \Delta_1)^d \geq \mathrm{vol}(\Delta_0)^{d-1} \cdot \mathrm{vol}(\Delta_1), \tag{3}$$

where Δ_0 and Δ_1 are two convex bodies in $\mathbb{R}^d$ and $\mathrm{vol}_{d-1,1}(\Delta_0, \Delta_1)$ is the mixed volume of index $(d-1, 1)$, which is equal to

$$\lim_{\varepsilon \to 0+} \frac{\mathrm{vol}(\Delta_0 + \varepsilon\Delta_1) - \mathrm{vol}(\Delta_0)}{\varepsilon d}.$$

We refer readers to the work of Minkowski [31] for the notion of mixed volumes in convex geometry. See [8, § 7.29] for more details.

Note that (3) is one of the inequalities of Alexandrov-Fenchel type for mixed volumes, which is actually equivalent to Brunn-Minkowski inequality (see for example [37, § 7.2] for a proof). Note that the above inequalities in convex geometry are similar to some inequalities of intersection numbers in algebraic geometry. By using toric varieties, Teissier [38] and Khovanskii [13, § 4.27] have given proofs of the Alexandrov-Fenchel inequality by using the Hodge index theorem.

In the arithmetic geometry setting, Bertrand [6, § 1.2] has established a lower bound for the height function on an arithmetic variety, and interpreted it as an arithmetic analogue of the isoperimetric inequality. In [15], the author has proposed the notion of positive intersection product in Arakelov geometry and proved an analogue of the isoperimetric inequality in the form of (3), by using the arithmetic Brunn-Minkowski inequality established by Yuan [40].

The purpose of this article is to propose a relative version of the arithmetic isoperimetric inequality and Brunn-Minkowski inequality as follows by taking into account the relative structure of arithmetic varieties with respect to an arithmetic curve. We refer to Theorems 3.5, 3.3, 4.1 and 4.5 for the proof and for various refined forms of the statement.

Theorem 1.1. — *Let K be a number field and X be a geometrically integral projective scheme of dimension $d \geq 1$ over* $\operatorname{Spec} K$. *Let $\overline{D}_0$ and $\overline{D}_1$ be nef* *adelic $\mathbb{R}$-Cartier divisors on X such that D_0 and D_1 are big. Then one has*

$$(4)\quad (d+1)\widehat{\deg}(\overline{D}_0^d\cdot\overline{D}_1)\geq d\left(\frac{\deg(D_1^d)}{\deg(D_0^d)}\right)^{1/d}\widehat{\deg}(\overline{D}_0^{d+1})+\frac{\deg(D_0^d)}{\deg(D_1^d)}\widehat{\deg}(\overline{D}_1^{d+1}).$$

If $(\overline{D}_i)_{i=1}^n$ is a family of nef adelic $\mathbb{R}$-Cartier divisors such that $D_1,\ldots,D_n$ are big, then one has

$$(5)\qquad \frac{\widehat{\deg}((\overline{D}_1+\cdots+\overline{D}_n)^{d+1})}{\deg((D_1+\cdots+D_n)^d)}\geq\varphi(D_1,\ldots,D_n)^{-1}\sum_{i=1}^n\frac{\widehat{\deg}(\overline{D}_i^{d+1})}{\deg(D_i^d)},$$

where

$$(6)\qquad \varphi(D_1,\ldots,D_n):=d+1-d\frac{\deg(D_1^d)^{1/d}+\cdots+\deg(D_n^d)^{1/d}}{\deg((D_1+\cdots+D_n)^d)^{1/d}}.$$

Compared to the direct arithmetic analogue of the Brunn-Minkowski inequality (see [40, Theorem B]), the inequality (5) distinguishes the contribution of the geometric structure of the $\mathbb{R}$-Cartier divisors $D_1,\ldots,D_n$.

In the particular case where $d=2$ (that is, where X is an arithmetic surface), the inequality (4) becomes a form of the arithmetic Hodge index inequality

$$2\widehat{\deg}(\overline{D}_0\cdot\overline{D}_1)\geq\frac{\deg(D_1)}{\deg(D_0)}\widehat{\deg}(\overline{D}_0^2)+\frac{\deg(D_0)}{\deg(D_1)}\widehat{\deg}(\overline{D}_1^2),$$

established in [17, Theorem 4.12], which is equivalent to the arithmetic Hodge index theorem of Faltings [19] and Hriljac [24], since the above inequality is equivalent to

$$\left(\frac{\overline{D}_0}{\deg(D_0)}-\frac{\overline{D}_1}{\deg(D_1)}\right)^2\leq 0.$$

We refer readers to [17, Corollary 4.14 and Remark 4.15] for a comparison of different forms of the statement. Similarly to [17], we also use the interpretation of the arithmetic self-intersection number of a nef and big adelic $\mathbb{R}$-Cartier divisor $\overline{D}$ as the integral of a concave function on the Okounkov body $\Delta(D)$ of the $\mathbb{R}$-Cartier divisor D, which is a convex body in $\mathbb{R}^d$. However the proof of Theorem 1.1 follows a strategy which is different from the way indicated in [17]. In fact, in [17] the author has introduced for any couple (Δ_1,Δ_2) of convex bodies in $\mathbb{R}^d$, a number $\rho(\Delta_1,\Delta_2)$ (called the correlation index of Δ_1 and Δ_2) which measures the degree of uniformity in the Minkowski sum $\Delta_1+\Delta_2$ of the sum of two uniform random variables[1] valued in Δ_1 and Δ_2, respectively.

1. For any convex body $\Delta\subset\mathbb{R}^d$, a Borel probability measure on $\mathbb{R}^d$ is called the *uniform distribution* on Δ if it is absolutely continuous with respect to the Lebesgue measure, and the corresponding Radon-Nikodym density is $1/\operatorname{vol}(\Delta)$, where $\operatorname{vol}(\Delta)$ is the Lebesgue measure of Δ; a random variable valued in $\mathbb{R}^d$ is said to be *uniformly distributed* in Δ if it follows this measure as its probability law.

The inequality

$$\frac{\widehat{\deg}((\overline{D}_1+\overline{D}_2)^{d+1})}{\mathrm{vol}(\Delta(D_1)+\Delta(D_2))}\geq\rho(\Delta(D_1),\Delta(D_2))^{-1}\left(\frac{\widehat{\deg}(\overline{D}_1^{d+1})}{\mathrm{vol}(\Delta(D_1))}+\frac{\widehat{\deg}(\overline{D}_2^{d+1})}{\mathrm{vol}(\Delta(D_2))}\right)$$

has been established for any couple $(\overline{D}_1,\overline{D}_2)$ of nef and big adelic $\mathbb{R}$-Cartier divisors on X, and it has been suggested that the estimation of the correlation index $\rho(\Delta(D_1),\Delta(D_2))$ should lead to more concrete inequalities which are similar to (5). However, the main point in this approach is to construct a suitable correlation structure between two random variables which are uniformly distributed in $\Delta(D_1)$ and $\Delta(D_2)$ such that the sum of the random variables is as uniform as possible in the Minkowski sum $\Delta(D_1)+\Delta(D_2)$. For example, we can deduce from a work of Bobkov and Madiman [7] the following uniform upper bound (where we choose independent random variables) (see [17, Proposition 2.9])

$$\rho(\Delta(D_1),\Delta(D_2))\leq\binom{2d}{d}.$$

This upper bound is larger than $\varphi(D_1,D_2)$, the latter being clearly bounded from above by $d+1$.

The strategy of this article is inspired by the works of Knothe [28] and Brenier [11, 12] on measure preserving diffeomorphisms between two convex bodies (see also the works of Gromov [22], Alesker, Dar and Milman [2] for more developments of this method and for applications in Alexandrov-Fenchel type inequalities in the convex geometry setting, and the memoir of Barthe [3] for diverse applications of this method in functional inequalities). Given a couple (Δ_0,Δ_1) of convex bodies in $\mathbb{R}^d$, one can construct a C^1 diffeomorphism $f:\Delta_0\to\Delta_1$ which transports the uniform probability measure of Δ_0 to that of Δ_1; that is, the determinant of the Jacobian J_f is constant on the interior of Δ_0. This diffeomorphism is not unique: in the construction of Knothe, the Jacobian J_f is upper triangular, while in the construction of Brenier, J_f is symmetric and positive definite.

If Z_0 is a random variable which is uniformly distributed in Δ_0, then $Z_1:=f(Z_0)$ is uniformly distributed in Δ_1. One may expect that the random variable Z_0+Z_1 follows a probability law which is close to the uniform probability measure on $\Delta_0+\Delta_1$. In fact, the random variable Z_0+Z_1 can also be expressed as $Z_0+f(Z_0)$. Its probability law identifies with the direct image of the uniform probability measure on Δ_0 by the map $\mathrm{Id}+f$, admitting $\mathrm{Id}+J_f$ as its Jacobian, the determinant of which can be estimated in terms of the determinant of J_f. In the case where $\mathrm{Id}+f$ is injective (for example the Knothe map), this lower bound leads to the following upper bound for the correlation index

$$\rho(\Delta_0,\Delta_1)\leq\frac{\mathrm{vol}(\Delta_0+\Delta_1)}{(\mathrm{vol}(\Delta_0)^{1/d}+\mathrm{vol}(\Delta_1)^{1/d})^d}. \tag{7}$$

By this method we obtain a weaker version of inequality (5) in the case where $n = 2$ by replacing $\varphi(D_1, D_2)$ with

$$\frac{\mathrm{vol}(D_1 + D_2)}{(\mathrm{vol}(D_1)^{1/d} + \mathrm{vol}(D_2)^{1/d})^d}.$$

This function is, in general, not bounded when D_1 and D_2 vary.

The main idea of the article is to use an infinitesimal variant of the above argument. Instead of considering the map $\mathrm{Id} + f : \Delta_0 \to \Delta_0 + \Delta_1$, we consider $\mathrm{Id} + \varepsilon f : \Delta_0 \to \Delta_0 + \varepsilon\Delta_1$ for $\varepsilon > 0$ sufficiently small, and use it to establish an isoperimetric inequality in an integral form as follows (see Theorem 3.1 *infra*).

Theorem 1.2. — *Let G_0 and G_1 be two Borel functions on Δ_0 and Δ_1, respectively. We assume that they are integrable with respect to the Lebesgue measure. Suppose given, for any $\varepsilon \in [0, 1]$, an almost everywhere non-negative Borel function H_ε on $\Delta_0 + \varepsilon\Delta_1$ such that*

$$\forall\,(x, y) \in \Delta_0 \times \Delta_1, \quad H_\varepsilon(x + \varepsilon y) \geq G_0(x) + \varepsilon G_1(y).$$

Then the following inequality holds

$$\liminf_{\varepsilon \to 0+} \frac{\int_{\Delta_0 + \varepsilon\Delta_1} H_\varepsilon(z)\mathrm{d}z - \int_{\Delta_0} G_0(x)\mathrm{d}x}{\varepsilon}$$
$$\geq d\left(\frac{\mathrm{vol}(\Delta_1)}{\mathrm{vol}(\Delta_0)}\right)^{1/d} \int_{\Delta_0} G_0(x)\mathrm{d}x + \frac{\mathrm{vol}(\Delta_0)}{\mathrm{vol}(\Delta_1)} \int_{\Delta_1} G_1(y)\mathrm{d}y.$$

By this method we obtain a relative form of the arithmetic isoperimetric inequality as in (4) and then deduce the arithmetic relative Brunn-Minkowski inequality (5) following the classic procedure of deducing the Brunn-Minkowski inequality from the isoperimetric inequality. Note that this does not signify that we improve inequality (7) by replacing the right-hand side of the inequality with

$$d + 1 - d\frac{\mathrm{vol}(\Delta_0)^{d/1} + \mathrm{vol}(\Delta_1)^{1/d}}{\mathrm{vol}(\Delta_0 + \Delta_1)^{1/d}}.$$

For example, it remains an open question to determine whether the correlation index $\rho(\Delta_0, \Delta_1)$ is always bounded from above by $d + 1$.

Finally, I would like to cite several refinements of the Brunn-Minkowski inequality in convex geometry, where the results are also expressed in a relative form similarly to (5), either with respect to an orthogonal projection on a hyperplane [23] or in terms of a comparison between the volume and the mixed volume [20] in the style of Bergstrom's inequality [4]. It is not excluded that the method presented in this article will bring new ideas to the research efforts in these directions.

The article is organized as follows. In the second section, we recall the notation and basic facts about adelic $\mathbb{R}$-Cartier divisors. In the third section, we