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POINTWISE CONVERGENCE FOR THE SCHRÖDINGER EQUATION
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1. Introduction: the Carleson problem

1.1. Solutions to the Schrödinger equation

Suitably normalised, the free Schrödinger equation on $\mathbb{R}^n$ is the second order partial differential equation

$$iu_t - \Delta_x u = 0. \tag{1}$$

Here u is a complex-valued function of the space-time variables $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, whilst u_t and $\Delta_x u$ denote the first order time derivative and spatial Laplacian, respectively. We are interested in the Cauchy problem for this equation, whereby we specify an initial datum f and wish to solve

$$\begin{cases} iu_t - \Delta_x u = 0, \\ u(x,0) = f(x) \end{cases} \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}. \tag{2}$$

Depending on our hypotheses on f, what it means for u to be a 'solution' to the equation (2) varies. Here we consider two examples:

Classical solution. If f is sufficiently regular, then elementary Fourier transform methods show that (2) has a unique solution in the classical sense.(1) For instance, if we assume $f \in \mathscr{S}(\mathbb{R}^n)$, the Schwartz space, then the unique solution is given by

$$u(x,t) := e^{it\Delta} f(x)$$

where $e^{it\Delta}$ is the *Schrödinger propagator*

$$e^{it\Delta} f(x) := \frac{1}{(2\pi)^n} \int_{\widehat{\mathbb{R}}^n} e^{i(x\cdot\xi + t|\xi|^2)} \hat{f}(\xi)\, \mathrm{d}\xi. \tag{3}$$

(1) In particular, the derivatives u_t and $\Delta_x u$ are all well-defined in the usual sense from calculus, and the identities in (2) hold pointwise.

Note that the regularity – or smoothness – of the initial datum f is crucial to these observations. Indeed, the smoothness of f directly translates into the decay of the Fourier transform $\hat{f}(\xi)$ as $|\xi| \to \infty$. This decay ensures the integral in (3) is well-defined and also allows one to pass the derivatives inside the integral in order to verify (1).

L^2 *solution*. Now suppose $f \in L^2(\mathbb{R}^n)$, without any additional regularity assumptions. In this case, Plancherel's theorem allows us to define the Fourier transform $\hat{f}$ as a function in $L^2(\widehat{\mathbb{R}}^n)$, but in general we cannot conclude that $\hat{f}$ is integrable. Consequently, the integral formula (3) is not well-defined in the classical sense.

To circumvent these issues, we further appeal to the L^2 theory of Fourier transform. Note that the propagator $e^{it\Delta}$ introduced above can be interpreted as a linear operator on $\mathscr{S}(\mathbb{R}^n)$ which, given an initial datum f, outputs the solution at time t. Using Plancherel's theorem, we can extend $e^{it\Delta}$ to a Fourier multiplier operator acting on the whole of $L^2(\mathbb{R}^n)$. In particular, we define

$$e^{it\Delta} f := \mathscr{F}^{-1}\big(e^{it|\,\cdot\,|^2} \cdot \mathscr{F} f\big) \qquad \text{for } f \in L^2(\mathbb{R}^n),$$

where here $\mathscr{F}$ denotes the Fourier transform acting on $L^2(\mathbb{R}^n)$. Furthermore, this operator is an isometry of the L^2 space, in the sense that

$$\|e^{it\Delta} f\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)} \qquad \text{for all } f \in L^2(\mathbb{R}^n) \text{ and all } t \in \mathbb{R}; \tag{4}$$

this identity is typically referred to as *conservation of energy*.

As before, we may define

$$u(x,t) := e^{it\Delta} f(x),$$

but in general this is no longer a classical solution to the Schrödinger equation: for instance, for a fixed time t, the best we can say about $u(\,\cdot\,, t)$ is that it belongs to $L^2(\mathbb{R}^n)$ and so the Laplacian $\Delta_x u$ is not defined in the classical sense. However, we can interpret u as a solution to (1) in the sense of distributions. Indeed, using (4) it is not difficult to show u defines a distribution in $\mathscr{S}'(\mathbb{R}^{n+1})$ and so $\partial_t u$ and $\Delta_x u$ can be understood in the distributional sense. Furthermore, a simple Fourier analytic argument shows $\langle i\partial_t u - \Delta_x u, \phi \rangle = 0$ for all test functions $\phi \in \mathscr{S}(\mathbb{R}^{n+1})$.

1.2. The Carleson problem

Once a solution u to (2) has been constructed, it is natural to investigate the behaviour of u and how it relates to the initial datum f. There is a huge variety of different questions one can ask in this direction. Here we are interested in the classical *Carleson problem*, which aims to understand whether the initial datum can be recovered as a pointwise limit of the solution.

First consider the case where $f \in \mathscr{S}(\mathbb{R}^n)$, so that the solution $u(x,t) := e^{it\Delta}f(x)$ is classically defined. By definition, we know the solution u satisfies $u(x,0) = f(x)$ and is differentiable, and therefore continuous, with respect to t. In particular,

$$\lim_{t \to 0_+} e^{it\Delta}f(x) = f(x) \qquad \text{for all } x \in \mathbb{R}^n. \tag{5}$$

The Carleson problem asks to what extent this elementary limit identity continues to hold when we consider more general L^2 solutions to the Schrödinger equation.

Since an L^2 function is only defined almost everywhere, in order to make sense of the problem for general initial data in $L^2(\mathbb{R}^n)$ it is necessary to weaken the requirement that convergence holds for *all* $x \in \mathbb{R}^n$ in (5) to *almost all* $x \in \mathbb{R}^n$. That is, given $f \in L^2(\mathbb{R}^n)$ we wish to determine whether

$$\lim_{t \to 0_+} e^{it\Delta}f(x) = f(x) \qquad \text{for almost every } x \in \mathbb{R}^n. \tag{6}$$

Nevertheless, it is still unclear how to precisely interpret the above limit, since for every time slice t (belonging to the *continuum* $[0,1]$, say) we have a choice of representation for $e^{it\Delta}f$. We shall gloss over these technicalities for now and return to them in §3.2 below.

It is not difficult to show that the limit holds in the L^2-sense: that is, given $f \in L^2(\mathbb{R}^n)$ we have

$$\lim_{t \to 0_+} \|e^{it\Delta}f - f\|_{L^2(\mathbb{R}^n)} = 0. \tag{7}$$

Indeed, this can be easily verified for $f \in \mathscr{S}(\mathbb{R}^n)$ using the integral formula (3) for the propagator and the dominated convergence theorem. One can then pass to general $f \in L^2(\mathbb{R}^n)$ via density, using the conservation of energy identity (4).

On the other hand, there are examples of $f \in L^2(\mathbb{R}^n)$ for which (6) in fact **fails** (see §1.3 below). Thus, we are interested in determining an additional hypothesis on f under which the above norm convergence (7) can be 'upgraded' to almost everywhere convergence. Contrasting the situation for $f \in \mathscr{S}(\mathbb{R}^n)$ with that for general $f \in L^2(\mathbb{R}^n)$, it is natural that the additional hypothesis should enforce some degree of regularity on the initial datum.

The above considerations lead us to consider the Sobolev spaces $H^s(\mathbb{R}^n)$. Roughly speaking, $H^s(\mathbb{R}^n)$ consists of all $f \in L^2(\mathbb{R}^n)$ with derivatives up to order s lying also in $L^2(\mathbb{R}^n)$. More precisely,

$$H^s(\mathbb{R}^n) := \big\{f \in L^2(\mathbb{R}^n) : (1 - \Delta_x)^{s/2}f \in L^2(\mathbb{R}^n)\big\}, \qquad s \geqslant 0,$$

where $(1 - \Delta)^{s/2}$ denotes the fractional differential operator, defined in terms of the Fourier transform $\mathscr{F}$ now acting on the space of distributions $\mathscr{S}'(\mathbb{R}^n)$ by

$$(1 - \Delta_x)^{s/2}f := \mathscr{F}^{-1}\big((1 + |\cdot|^2)^{s/2} \cdot \mathscr{F}f\big).$$

In particular, given $f \in L^2(\mathbb{R}^n)$, we can always make sense of the fractional derivative $(1-\Delta_x)^{s/2} f$ as a distribution, and $f \in H^s(\mathbb{R}^n)$ if this distribution coincides with an L^2 function. It is clear from the definitions that

$$H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n) \quad \text{and} \quad H^{s_1}(\mathbb{R}^n) \supseteq H^{s_2}(\mathbb{R}^n) \quad \text{for } 0 \leqslant s_1 \leqslant s_2.$$

Sobolev spaces provide a natural framework in which to formalise the Carleson problem.

Problem 1.1 (Carleson, 1980). *Determine the values of $s \geqslant 0$ such that*

$$\text{if } f \in H^s(\mathbb{R}^n), \quad \text{then} \quad \lim_{t \to 0} e^{it\Delta} f(x) = f(x) \quad \text{for almost every } x \in \mathbb{R}^n. \tag{8}$$

That is, we wish to determine the minimal degree of regularity (measured in terms of the Sobolev space index s) for which almost everywhere convergence is guaranteed to hold.

Aside from its intrinsic appeal, Problem 1.1 is intimately related to important questions regarding the distribution of the solution $e^{it\Delta} f(x)$ in space-time. Pointwise convergence is typically proved via analysis of the *Schrödinger maximal operator*, an object of interest in its own right. The maximal operator can in turn be studied using *fractal energy estimates* for the Schrödinger solutions. We introduce these concepts in §3.2 and §3.5 below. Through these connections, progress on Problem 1.1 has led to new developments on a surprising array of different problems, such as the Falconer distance problem (see, for instance, Du and Zhang, 2019; Guth, Iosevich, et al., 2020) and the Fourier restriction conjecture (see Wang and Wu, 2022).

1.3. A resolution of the Carleson problem: introducing the key results

Problem 1.1 has a rich history, paralleling many important developments in harmonic analysis over the last 40 years. We do not intend to give a complete survey of the relevant literature, but focus on definitive results and recent highlights.

Whilst the $n = 1$ case of Problem 1.1 was fully understood by the early 1980s through the works of Carleson (1980) and Dahlberg and Kenig (1982), in higher dimensions the situation is much more nuanced. Nevertheless, a recent series of dramatic developments brought about an almost complete resolution.

Necessary conditions. — Problem 1.1 splits into two parts: finding necessary conditions for the index s for (8) to hold and finding sufficient conditions. Both parts are difficult. The recent spate of activity on the Carleson problem was initiated by the surprising discovery of a new necessary condition on s.

Theorem 1.2 (Bourgain, 2016). *For all $s < \frac{n}{2(n+1)}$, there exists some $f \in H^s(\mathbb{R}^n)$ such that (6) fails.*

Theorem 1.2 relies on the construction of an explicit[(2)] initial datum f; the proof is intricate, involving number theoretic considerations. Prior to Bourgain (2016), weaker necessary conditions were established in Bourgain (2013b), Dahlberg and Kenig (1982), and Lucà and Rogers (2017).

We shall not discuss the proof of Theorem 1.2 here, but instead refer the reader to the detailed exposition in Pierce (2020). An alternative argument, based on ergodic arguments rather than number theory, can also be found in Lucà and Rogers (2019).

Sufficient conditions. — We now turn to positive results, which form the focus of this article. In the wake of Bourgain's counterexample, there was a flurry of activity on the Carleson problem. In a major advance, the $n = 2$ case was completely settled through work of Du, Guth, and Li (2017). The higher dimensional case later followed in a landmark paper of Du and Zhang (2019).

Theorem 1.3 (Du and Zhang, 2019[(3)]). *If $f \in H^s(\mathbb{R}^n)$ for some $s > \frac{n}{2(n+1)}$, then*

$$\lim_{t \to 0_+} e^{it\Delta} f(x) = f(x) \qquad \textit{holds for almost every } x \in \mathbb{R}^n.$$

Remark 1.4. As previously noted, there are measure-theoretic technicalities regarding the meaning of the above statement, since the limit is taken over a continuum. We address this in §3.2 below.

Together, Theorem 1.2 and Theorem 1.3 give an almost complete[(4)] answer to the Carleson problem and constitute a major milestone in harmonic analysis and PDE. Furthermore, Theorem 1.3 is in fact a special case of a significantly more general result proved in Du and Zhang (2019), which has a variety of additional applications: see §3.5 below.

The proof of Theorem 1.3 builds on many important developments in harmonic analysis and previous works on the Carleson problem in particular. For the purpose of this article, we shall roughly divide the recent history of the problem into two epochs.

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(2) Strictly speaking, the proof of Theorem 1.2 proceeds by constructing a counterexample to the $H^s(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ boundedness of the *Schrödinger maximal operator*. This in turn implies the existence of a counterexample to (8) through a variant of Stein's maximal principle. See §3.2.

(3) The $n = 1$ and $n = 2$ cases of Theorem 1.3 were established earlier in Carleson (1980) and Du, Guth, and Li (2017), respectively.

(4) That is, except for the question of behaviour at the endpoint exponent $s = n/(2(n+1))$.