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ROTATION INVARIANCE OF CRITICAL PLANAR PERCOLATION
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by **Vincent Tassion**

Introduction

Consider critical independent percolation on the square lattice $\mathbb{Z}^2$, viewed as a graph: For each edge, flip a coin, the edge is kept with probability $p = 1/2$, it is deleted otherwise. We thus obtain a random subgraph of $\mathbb{Z}^2$. The distribution of this random graph is invariant under rotation of angle $\pi/2$, as it inherits the symmetries of the lattice. But if we consider the large connected components, new symmetries emerge: Duminil-Copin et al. (2020) have shown that the distribution of these connected components is asymptotically invariant under all rotations. This result represents major progress towards understanding critical phenomena in planar statistical mechanics. The main conjecture in the field is that the distribution of large connected components is in fact invariant by conformal transformations, and it satisfies a principle of universality: this distribution does not depend on the underlying lattice. In this article, we give some general background on Bernoulli percolation, we state the new rotation invariance result and discuss some key aspects of it: what role does the parameter 1/2 play? What heuristic reasons justify the emergence of these symmetries? What are the main ideas behind rotational invariance? We mainly focus on one important ingredient of the proof: the star-triangle transformation. Originated from the study of electrical networks, it allows the authors to relate percolation on the square lattice to other auxiliary graphs, and "import" extra symmetries satisfied by these graphs (namely symmetry under reflections).

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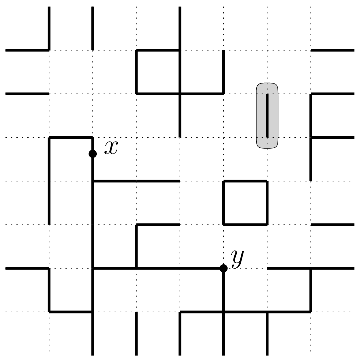
1. Phase transition of Bernoulli percolation

Bernoulli percolation was introduced in 1957 by BROADBENT and HAMMERSLEY (1957) in order to understand the propagation of a fluid in a porous medium, modeled as follows. Consider the square lattice $\mathbb{Z}^2$, which we see as a planar graph embedded in the complex plane: its vertex set is $V = \{u + \mathrm{i}v \,:\, u, v \in \mathbb{Z}\}$, and the edge set E is given by all linear segments $[u, v]$ with $|u - v| = 1$. Fix a parameter $p \in [0, 1]$, which represents the porosity of the material we want to model.

For each edge $e \in E$ toss a biased coin, and define

$$\omega_e = \begin{cases} 0 & \text{with probability } 1 - p, \\ 1 & \text{with probability } p, \end{cases}$$

independently of the other edges. We say that the edge e is open if $\omega_e = 1$ (solid edges in the figure below) and closed if $\omega_e = 0$ (dotted edges).



The terminology open/closed comes from the interpretation of ω as a porous material: the fluid can only travel through open edges, and percolation aims at describing the different paths that the fluid can follow. To this end, it is convenient to identify ω with the union of all the open edges. This way, we see ω as a closed subset of $\mathbb{C}$ and define its corresponding topological properties. We call open path a continuous path with support in ω. For example, in the picture above, there exists an open path from x to y. We emphasize that we do not impose that the path starts and ends at vertices of $\mathbb{Z}^2$. We call cluster a connected component of ω. For example, above, we surrounded a cluster made of a single edge. Despite this elementary mathematical description, Bernoulli percolation offers a natural probabilistic framework to develop and understand the theory of phase transitions, a key notion in statistical mechanics.

A natural question for Bernoulli percolation is whether there exists an infinite cluster in ω. The answer depends on the underlying parameter: if $p = 0$ we have

$\omega = \emptyset$ and there is no infinite cluster. For $p = 1$ all the edges are open, and there is a unique infinite cluster. When p varies continuously from 0 to 1, we observe a drastic change of behaviour at a certain critical value p_c. More precisely, elementary monotonicity and ergodic arguments show that there exists a critical parameter p_c such that

$$\begin{aligned} p < p_c \quad &\Longrightarrow \quad \text{all the clusters are finite almost surely,} \\ p > p_c \quad &\Longrightarrow \quad \text{there exists an infinite cluster almost surely.} \end{aligned}$$

In a groundbreaking work, Kesten (1980) proved that $p_c = 1/2$ for Bernoulli percolation on the square lattice and obtained a precise description of the subcritical phase ($p < p_c$) and the supercritical phase ($p > p_c$). The behaviour at $p = p_c = 1/2$ is still the object of famous conjectures in the field, and the present article reviews some recent progress in the study of this critical regime.

We refer to the manuscripts of Grimmett (1999), Bollobás and Riordan (2006) and Werner (2009) for general background on percolation theory.

Organization of this article. In Section 2, we state the new rotation invariance result of Duminil-Copin et al. (2020), and explain its relation to conformal invariance and universality of planar percolation in Section 3. The proof of rotation invariance relies on a discrete tool, the star-triangle transformation. In Section 4, we introduce this transformation, and in Section 5 we explain how it can be used to study the symmetries of certain percolation quantities. In Section 6, we discuss the role of the embedding of the graph and explain how the proof reduces to a key stability lemma.

2. Crossing probabilities and rotation invariance

In this section, we consider critical Bernoulli percolation at $p = p_c = 1/2$ and we discuss the rotation invariance result of Duminil-Copin et al. (2020). To keep this presentation light, we state a weaker version of the result: first we restrict to Bernoulli percolation, while the original result applies to more general models (FK percolation). Second, we state it in terms of rectangle crossings: the original result states that the whole collection of clusters is rotationally invariant, after a suitable truncation. Stating this strong result would require more background, in particular a careful definition of the state space for the collection of clusters.

For every a, b such that $0 \leqslant a \leqslant b$, we define the rectangle

$$R_{a,b} = [-a, a] \times [-b, b].$$

Through this article we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. In particular, we see $R_{a,b}$ as a subset of $\mathbb{C}$. Let ω be a critical Bernoulli percolation of the plane, seen as a random closed subset of $\mathbb{C}$. We say that $R_{a,b}$ is crossed in ω if there exists an open path in $R_{a,b} \cap \omega$ from the left side $\{-a\} \times [-b, b]$ to the right side $\{a\} \times [-b, b]$. We write $R^{\theta}_{a,b}$ for the rotation of $R_{a,b}$ with angle θ around 0, and say that $R^{\theta}_{a,b}$ is crossed in ω if there exists an open path in $R^{\theta}_{a,b} \cap \omega$ connecting the images (under the θ-rotation) of the left and right sides of $R_{a,b}$. See Figure 1 for an illustration of this event. We emphasize that the connection probabilities are defined in terms of continuous subsets of the plane, hence the crossing events are well defined for arbitrary real numbers a, b, θ.

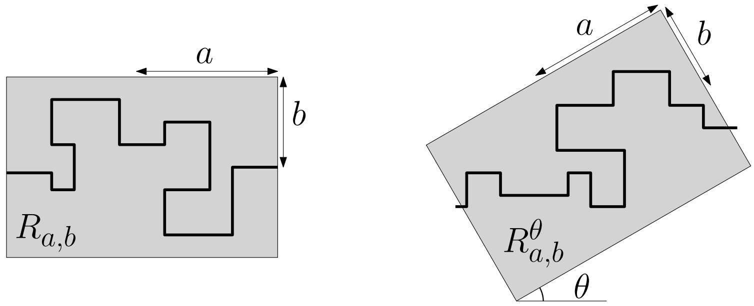


Figure 1: Diagrammatic representations of the events that $R_{a,b} = R^{0}_{a,b}$ is crossed (left) and $R^{\theta}_{a,b}$ is crossed with an arbitrary angle θ (right). In both cases, the solid path represents an open path connecting the left side to the right side of the rectangle.

Russo (1978), Seymour and Welsh (1978) proved that crossing probabilities with a fixed aspect ratio are non degenerated: For every fixed λ, θ, there exists $c > 0$ such that

$$\forall n \geqslant 1 \qquad c \leqslant \mathbb{P}[R^{\theta}_{\lambda n,n} \text{ is crossed in } \omega] \leqslant 1 - c.$$

The asymptotic behaviour of the critical crossing probabilities is not yet rigorously understood, and is the object of a major open problem (see e.g. Langlands, Pichet, et al., 1992), that we can state as follows.

Conjecture 2.1. *Consider a Bernoulli percolation ω on the square lattice with parameter $p = p_c = 1/2$.*

(i) *For every $\lambda \geqslant 1$, $\theta \in [0, \pi/2]$, the sequence $(\mathbb{P}[R^{\theta}_{\lambda n,n}$ is crossed in $\omega])_{n \geqslant 1}$ converges as n tends to infinity.*

(ii) *For every $\theta \in [0, \pi/2]$,*

$$\lim_{n\to\infty} P[R^{\theta}_{\lambda n,n} \textit{ is crossed in } \omega] = \lim_{n\to\infty} P[R_{\lambda n,n} \textit{ is crossed in } \omega].$$

The first part of the conjecture can be interpreted as a "dilatation invariance" of the model: the rectangle $R^{\theta}_{\lambda n,n}$ is a dilatation of the rectangle $R^{\theta}_{\lambda,1}$ by a factor n, and the crossing probabilities for large rectangles do not depend on the dilatation parameter n. The second part corresponds to a rotation invariance: the crossing probabilities for large rectangles do not depend on the angle θ of the rectangle.

Three years ago, Duminil-Copin et al. (2020) proved that crossing probabilities are invariant under rotation (which corresponds to the second item of the conjecture above). More precisely they establish the following theorem.

Theorem 2.2 (Duminil-Copin et al., 2020). *Consider a Bernoulli percolation ω on the square lattice with parameter $p = p_c = 1/2$. For every $\lambda \geqslant 1$ and every rotation angle $\theta \in [0, \pi/2]$, we have*

$$\mathbb{P}[R^{\theta}_{\lambda n,n} \text{ is crossed in } \omega] = \mathbb{P}[R_{\lambda n,n} \text{ is crossed in } \omega](1 + o(1))$$

as n tends to infinity.

Remarks:

- ▷ The case $\theta = \frac{\pi}{2}$ is easy because the lattice is already invariant under $\pi/2$-rotation. In contrast, the invariance for $\theta \in (0, \pi/2)$ is nontrivial and can not be deduced from the symmetries of the lattice.
- ▷ A self duality argument (see e.g. Grimmett, 1999) implies that the rectangles of the form $[0, n+1] \times [0, n]$ are crossed with probability $1/2$. Therefore, a direct corollary of Theorem 2.2 is that for every $\theta \in [0, \pi/2]$,

$$\lim_{n\to\infty} \mathbb{P}[R^{\theta}_{n,n} \text{ is crossed in } \omega] = \frac{1}{2}.$$

- ▷ The theorem does not state that the crossing probabilities converge and the first item in Conjecture 2.1 is still open.

3. Conformal invariance and universality

A much stronger symmetry of the crossing probabilities is conjectured, namely they are expected to be conformally invariant (see Langlands, Pouliot, and Saint-Aubin, 1994 and references therein). To state the conjecture, we use the notion of conformal rectangles, that we now define. Let $\lambda \geqslant 1$. We call conformal rectangle of modulus λ a pair (Ω, ϕ), where $\Omega \subset \mathbb{C}$ is a simply connected open set, and ϕ is a homeomorphism from the rectangle $R_{\lambda,1}$ to $\overline{\Omega}$ such that its restriction $\phi|_{(0,\lambda)\times(0,1)}$ is a conformal map from $(0, \lambda) \times (0, 1)$ to Ω.

For $n \geqslant 1$, notice that the blown up $(n \cdot \Omega, n \cdot \phi)$ is also a conformal rectangle of modulus λ, and in particular it has well-defined left and right sides. We say that $n \cdot \Omega$ is crossed if there exists an open path in $n \cdot \Omega$ from its left to its right side.