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THE STRONG ASYMPTOTIC FREENESS OF HAAR AND DETERMINISTIC MATRICES

BY Benoît COLLINS AND Camille MALE

ABSTRACT. – In this paper, we are interested in sequences of q-tuple of $N \times N$ random matrices having a strong limiting distribution (i.e., given any non-commutative polynomial in the matrices and their conjugate transpose, its normalized trace and its norm converge). We start with such a sequence having this property, and we show that this property pertains if the q-tuple is enlarged with independent unitary Haar distributed random matrices. Besides, the limit of norms and traces in non-commutative polynomials in the enlarged family can be computed with reduced free product construction. This extends results of one author (C. M.) and of Haagerup and Thorbjørnsen. We also show that a p-tuple of independent orthogonal and symplectic Haar matrices have a strong limiting distribution, extending a recent result of Schultz. We mention a couple of applications in random matrix and operator space theory.

RÉSUMÉ. – Dans cet article, nous nous intéressons au q-tuple de matrices $N \times N$ qui ont une distribution limite forte (i.e., pour tout polynôme non commutatif en les matrices et leurs adjoints, sa trace normalisée et sa norme convergent). Nous partons d'une telle suite de matrices aléatoires et montrons que cette propriété persiste si on rajoute au q-tuple des matrices indépendantes unitaires distribuées suivant la mesure de Haar. Par ailleurs, la limite des normes et des traces en des polynômes non commutatifs en la suite élargie peut être calculée avec la construction du produit libre réduit. Ceci étend les résultats d'un des auteurs (C.M.) et de Haagerup et Thorbjørnsen. Nous montrons aussi qu'un p-tuple de matrices indépendantes orthogonales et symplectiques a une distribution limite forte, étendant par là-même un résultat de Schultz. Nous passons aussi en revue quelques applications de notre résultat aux matrices aléatoires et à la théorie des espaces d'opérateur.

1. Introduction and statement of the main results

Following random matrix notation, we call GUE the Gaussian Unitary Ensemble, i.e., any sequence $(X_N)_{N\geqslant 1}$ of random variables where X_N is an $N \times N$ selfadjoint random matrix whose distribution is proportional to the measure $\exp\big(-N/2\mathrm{Tr}(A^2)\big)\mathrm{d}A$, where $\mathrm{d}A$ denotes the Lebesgue measure on the set of $N \times N$ Hermitian matrices. We call a unitary Haar matrix of size N any random matrix distributed according to the Haar measure on the compact group of N by N unitary matrices.

We recall for readers' convenience the following definitions from free probability theory (see [4, 20]).

Definition 1.1. – 1. *A* $\mathcal{C}^*$-probability space $(\mathcal{A}, .^*, \tau, \|\cdot\|)$ *consists of a unital* C^**-algebra* $(\mathcal{A}, .^*, \|\cdot\|)$ *endowed with a state* τ*, i.e., a linear map* $\tau\colon \mathcal{A} \to \mathbb{C}$ *satisfying* $\tau[\mathbf{1}_{\mathcal{A}}] = 1$ *and* $\tau[aa^*] \geqslant 0$ *for all* a *in* $\mathcal{A}$*. In this paper, we always assume that* τ *is a trace, i.e., that it satisfies* $\tau[ab] = \tau[ba]$ *for every* a, b *in* $\mathcal{A}$*. An element of* $\mathcal{A}$ *is called a (noncommutative) random variable. A trace is said to be* faithful *if* $\tau[aa^*] > 0$ *whenever* $a \neq 0$*. If* τ *is faithful, then for any* a *in* $\mathcal{A}$*,*

$$\|a\| = \lim_{k\to\infty} \Big(\tau\big[(a^*a)^k\big]\Big). \tag{1.1}$$

2. *Let* $\mathcal{A}_1, \ldots, \mathcal{A}_k$ *be* $*$*-subalgebras of* $\mathcal{A}$ *having the same unit as* $\mathcal{A}$*. They are said to be* free *if for all* $a_i \in \mathcal{A}_{j_i}$ *(*$i = 1, \ldots, k$*,* $j_i \in \{1, \ldots, k\}$*) such that* $\tau[a_i] = 0$*, one has*

$$\tau[a_1 \cdots a_k] = 0$$

as soon as $j_1 \neq j_2$*,* $j_2 \neq j_3, \ldots, j_{k-1} \neq j_k$*. Collections of random variables are said to be free if the unital subalgebras they generate are free.*

3. *Let* $\mathbf{a} = (a_1, \ldots, a_k)$ *be a* k*-tuple of random variables. The joint distribution of the family* $\mathbf{a}$ *is the linear form* $P \mapsto \tau\big[P(\mathbf{a}, \mathbf{a}^*)\big]$ *on the set of polynomials in* $2k$ *noncommutative indeterminates. By* convergence in distribution*, for a sequence of families of variables* $(\mathbf{a}_N)_{N\geqslant 1} = (a_1^{(N)}, \ldots, a_k^{(N)})_{N\geqslant 1}$ *in* $\mathcal{C}^*$*-algebras* $\big(\mathcal{A}_N, .^*, \tau_N, \|\cdot\|\big)$*, we mean the pointwise convergence of the map*

$$P \mapsto \tau_N\big[P(\mathbf{a}_N, \mathbf{a}_N^*)\big],$$

and by strong convergence in distribution*, we mean convergence in distribution, and pointwise convergence of the map*

$$P \mapsto \big\|P(\mathbf{a}_N, \mathbf{a}_N^*)\big\|.$$

4. *A family of noncommutative random variables* $\mathbf{x} = (x_1, \ldots, x_p)$ *is called a* free semicircular system *when the noncommutative random variables are free, selfadjoint (*$x_i = x_i^*$*,* $i = 1, \ldots, p$*), and for all* k *in* $\mathbb{N}$ *and* $i = 1, \ldots, p$*, one has*

$$\tau[x_i^k] = \int t^k d\sigma(t),$$

with $d\sigma(t) = \frac{1}{2\pi}\sqrt{4-t^2}\,\mathbf{1}_{|t|\leqslant 2}\,dt$ *the semicircle distribution.*

5. *A noncommutative random variable* u *is called a* Haar unitary *when it is unitary (*$uu^* = u^*u = \mathbf{1}_{\mathcal{A}}$*) and for all* n *in* $\mathbb{N}$*, one has*

$$\tau[u^n] = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

In their seminal paper [14], Haagerup and Thorbjørnsen proved the following result.

THEOREM 1.2 ([14] The strong asymptotic freeness of independent GUE matrices)
For any integer $N \geqslant 1$, let $X_1^{(N)}, \dots, X_p^{(N)}$ be $N \times N$ independent GUE matrices and let $(x_1, \dots, x_p)$ be a free semicircular system in a $\mathcal{C}^*$-probability space with faithful state. Then, almost surely, for all polynomials P in p noncommutative indeterminates, one has

$$\big\|P(X_1^{(N)}, \dots, X_p^{(N)})\big\| \underset{N\to\infty}{\longrightarrow} \big\|P(x_1, \dots, x_p)\big\|,$$

where $\|\cdot\|$ denotes the operator norm in the left hand side and the norm of the $\mathcal{C}^*$-algebra in the right hand side.

This theorem is a very deep result in random matrix theory, and had an important impact. Firstly, it had significant applications to C^*-algebra theory [14, 21], and more recently to quantum information theory [5, 8]. Secondly, it was generalized in many directions. Schultz [24] has shown that Theorem 1.2 is true when the GUE matrices are replaced by matrices of the Gaussian Orthogonal Ensemble (GOE) or by matrices of the Gaussian Symplectic Ensemble (GSE). Capitaine and Donati-Martin [6] and, very recently, Anderson [3] have shown the analogue for certain Wigner matrices.

Another significant extension of Haagerup and Thorbjørnsen's result was obtained by one author (C. M.) in [18], where he showed that if in addition to independent GUE matrices, one also has an extra family of independent matrices with strong limiting distribution, the result still holds.

THEOREM 1.3 ([18] The strong asymptotic freeness of $\mathbf{X}_N, \mathbf{Y}_N$)
For any integer $N \geqslant 1$, we consider

- *a p-tuple $\mathbf{X}_N$ of $N \times N$ independent GUE matrices,*
- *a q-tuple $\mathbf{Y}_N$ of $N \times N$ matrices, possibly random but independent of $\mathbf{X}_N$.*

The above random matrices live in the $\mathcal{C}^*$-probability space $(\mathrm{M}_N(\mathbb{C}), .^*, \tau_N, \|\cdot\|)$, where τ_N is the normalized trace on the set $\mathrm{M}_N(\mathbb{C})$ of $N \times N$ matrices. In a $\mathcal{C}^*$-probability space $(\mathcal{A}, .^*, \tau, \|\cdot\|)$ with faithful trace, we consider

- *a free semicircular system $\mathbf{x}$ of p variables,*
- *a q-tuple $\mathbf{y}$ of noncommutative random variables, free from $\mathbf{x}$.*

If $\mathbf{y}$ is the strong limit in distribution of $\mathbf{Y}_N$, then $(\mathbf{x}, \mathbf{y})$ is the strong limit in distribution of $(\mathbf{X}_N, \mathbf{Y}_N)$.

In other words, if we assume that almost surely, for all polynomials P in $2q$ noncommutative indeterminates, one has

$$\tau_N\big[P(\mathbf{Y}_N, \mathbf{Y}_N^*)\big] \underset{N\to\infty}{\longrightarrow} \tau\big[P(\mathbf{y}, \mathbf{y}^*)\big], \tag{1.2}$$

$$\big\|P(\mathbf{Y}_N, \mathbf{Y}_N^*)\big\| \underset{N\to\infty}{\longrightarrow} \big\|P(\mathbf{y}, \mathbf{y}^*)\big\|, \tag{1.3}$$

then, almost surely, for all polynomials P in $p+2q$ noncommutative indeterminates, one has

$$\tau_N\big[P(\mathbf{X}_N, \mathbf{Y}_N, \mathbf{Y}_N^*)\big] \underset{N\to\infty}{\longrightarrow} \tau\big[P(\mathbf{x}, \mathbf{y}, \mathbf{y}^*)\big], \tag{1.4}$$

$$\big\|P(\mathbf{X}_N, \mathbf{Y}_N, \mathbf{Y}_N^*)\big\| \underset{N\to\infty}{\longrightarrow} \big\|P(\mathbf{x}, \mathbf{y}, \mathbf{y}^*)\big\|. \tag{1.5}$$

The convergence in distribution, stated in (1.4), is the content of Voiculescu's asymptotic freeness theorem. We refer to [4, Theorem 5.4.10] for the original statement and for a proof. An alternative way to state (1.5) is the following interversion of limits: for any matrix $H_N = P(\mathbf{X}_N, \mathbf{Y}_N, \mathbf{Y}_N^*)$, where P is a fixed polynomial, if we denote $h = P(\mathbf{x}, \mathbf{y}, \mathbf{y}^*)$, then by the definition of the norm in terms of the state (1.1),

$$\lim_{N\to\infty} \lim_{k\to\infty} \Big(\tau_N \big[(H_N^* H_N)^k \big] \Big)^{\frac{1}{2k}} = \lim_{k\to\infty} \Big(\tau \big[(h^* h)^k \big] \Big)^{\frac{1}{2k}}.$$

It is natural to wonder whether, instead of GUE matrices, the same property holds for unitary Haar matrices. The main result of this paper is the following theorem.

THEOREM 1.4 (The strong asymptotic freeness of $U_1^{(N)}, \dots, U_p^{(N)}, \mathbf{Y}_N$)
For any integer $N \geqslant 1$, we consider

- *a p-tuple $\mathbf{U}_N$ of $N \times N$ independent unitary Haar matrices,*
- *a q-tuple $\mathbf{Y}_N$ of $N \times N$ matrices, possibly random but independent of $\mathbf{U}_N$.*

In a $\mathcal{C}^*$-probability space $(\mathcal{A}, .^*, \tau, \|\cdot\|)$ with faithful trace, we consider

- *a p-tuple $\mathbf{u}$ of free Haar unitaries,*
- *a q-tuple $\mathbf{y}$ of noncommutative random variables, free from $\mathbf{u}$.*

If $\mathbf{y}$ is the strong limit in distribution of $\mathbf{Y}_N$, then $(\mathbf{u}, \mathbf{y})$ is the strong limit in distribution of $(\mathbf{U}_N, \mathbf{Y}_N)$.

In order to solve this problem, it looks at first sight natural to attempt to mimic the proof of [14] and write a Master equation in the case of unitary matrices. However, even though such an identity can be obtained for unitary matrices, it is very difficult to manipulate it in the spirit of [14] in order to obtain the desired norm convergence. Part of the problem is that the unitary matrices are not selfadjoint, unlike the GUE matrices considered in [14], and in this context the linearization trick and the identities do not seem to fit well together. In order to bypass this problem, in this paper, we take a completely different route by building on Theorem 1.3 and using a series of folklore facts of classical probability and random matrix theory.

Our method applies to prove the strong convergence in distribution of Haar matrices on the orthogonal and the symplectic groups by building on the result of Schultz [24], which is the analogue of Theorem 1.2 for GOE or GSE matrices instead of GUE matrices. The analogue of Theorem 1.3 does not exist yet. If one shows that the estimates of matrix valued Stieltjes transforms in [18] can always be performed with the additional terms in the estimate of [24], then, following the lines of this paper, one gets Theorem 1.3 for Haar matrices on the orthogonal and the symplectic groups, instead of the unitary group only. Therefore, in the following Theorem, we stick to proving the strong convergence of independent unitary, orthogonal or symplectic Haar matrices, without "constant" matrices $\mathbf{Y}$:

THEOREM 1.5 (The strong asymptotic freeness of independent Haar matrices)
For any integer $N \geqslant 1$, let $U_1^{(N)}, \dots, U_p^{(N)}$ be a family of independent Haar matrices of one of the three classical groups. Let $u_1, \dots, u_p$ be free Haar unitaries in a $\mathcal{C}^*$-probability space with faithful state. Then, almost surely, for all polynomials P in $2p$ noncommutative indeterminates, one has

$$\big\| P(U_1^{(N)}, \dots, U_p^{(N)}, U_1^{(N)*}, \dots, U_p^{(N)*}) \big\| \underset{N\to\infty}{\longrightarrow} \big\| P(u_1, \dots, u_p, u_1^*, \dots, u_p^*) \big\|,$$