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ZERO-CYCLES ON VARIETIES OVER p-ADIC FIELDS AND BRAUER GROUPS

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ABSTRACT. – In this paper, we study the Brauer-Manin pairing of smooth proper varieties over a p-adic field, and determine the p-adic part of the image of the induced cycle map. We also compute A_0 of a potentially rational surface which splits over a wildly ramified extension.

RÉSUMÉ. – Dans cet article, nous étudions l'accouplement de Brauer-Manin des variétés propres et lisses sur un corps p-adique, et déterminons la partie p-adique de l'image de l'application cycle induite. Nous calculons aussi le A_0 d'une surface potentiellement rationnelle déployée sur une extension sauvagement ramifiée.

1. Introduction

Let k be a p-adic local field, and let X be a proper smooth geometrically integral variety over k. Let $\mathrm{CH}_0(X)$ be the Chow group of 0-cycles on X modulo rational equivalence. An important tool to study $\mathrm{CH}_0(X)$ is the natural pairing due to Manin [27]

$$\text{(M)} \qquad \mathrm{CH}_0(X) \times \mathrm{Br}(X) \longrightarrow \mathbb{Q}/\mathbb{Z},$$

where $\mathrm{Br}(X)$ denotes the Grothendieck-Brauer group $H^2_{\text{ét}}(X, \mathbb{G}_{\mathrm{m}})$. When $\dim(X) = 1$, using the Tate duality theorem for abelian varieties over p-adic local fields, Lichtenbaum [25] proved that (M) is non-degenerate and induces an isomorphism

$$\text{(L)} \qquad \mathrm{A}_0(X) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{Br}(X)/\mathrm{Br}(k), \mathbb{Q}/\mathbb{Z}).$$

Here $\mathrm{Br}(X)/\mathrm{Br}(k)$ denotes the cokernel of the natural map $\mathrm{Br}(k) \to \mathrm{Br}(X)$, and $\mathrm{A}_0(X)$ denotes the subgroup of $\mathrm{CH}_0(X)$ generated by 0-cycles of degree 0. An interesting question is as to whether the pairing (M) is non-degenerate when $\dim(X) \geq 2$. See [31] for surfaces with non-zero left kernel. See [45] for varieties with trivial left kernel. In this paper, we are concerned with the right kernel of (M) in the higher-dimensional case.

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1.1. – We assume that X has a regular model $\mathscr{X}$ which is proper flat of finite type over the integer ring $\mathfrak{o}_k$ of k. It is easy to see that the pairing (M) induces homomorphisms

$$\mathrm{CH}_0(X) \longrightarrow \mathrm{Hom}(\mathrm{Br}(X)/\mathrm{Br}(\mathscr{X}), \mathbb{Q}/\mathbb{Z}), \tag{1.1.1}$$

$$\mathrm{A}_0(X) \longrightarrow \mathrm{Hom}(\mathrm{Br}(X)/\mathrm{Br}(k) + \mathrm{Br}(\mathscr{X}), \mathbb{Q}/\mathbb{Z}), \tag{1.1.2}$$

where $\mathrm{Br}(X)/\mathrm{Br}(k)+\mathrm{Br}(\mathscr{X})$ denotes the quotient of $\mathrm{Br}(X)$ by the image of $\mathrm{Br}(k)\oplus\mathrm{Br}(\mathscr{X})$. If $\dim(X) = 1$, then $\mathrm{Br}(\mathscr{X})$ is zero, and the map (1.1.2) is the same as (L) (cf. [9] 1.7 (c)). Our main result is the following:

THEOREM 1.1.3. – *Assume that the purity of Brauer groups holds for* $\mathscr{X}$ (*see Definition* 2.1.1 *below*). *Then*:

(1) *The right kernel of the pairing* (M) *is exactly* $\mathrm{Br}(\mathscr{X})$*, that is, the map* (1.1.1) *has dense image with respect to the natural pro-finite topology on the right hand side.*
(2) *The map* (1.1.2) *is surjective.*

Restricted to the prime-to-p part, the assertion (1) is due to Colliot-Thélène and Saito [10]. The assertion (2) gives an affirmative answer to [6] Conjecture 1.4 (c), assuming the purity of Brauer groups, which holds if $\dim(\mathscr{X}) \leq 3$ or if $\mathscr{X}$ has good or semistable reduction (cf. Remark 2.1.2 below). Roughly speaking, Theorem 1.1.3 (1) asserts that if an element $\omega \in \mathrm{Br}(X)$ ramifies along the closed fiber of $\mathscr{X}$, then there exists a closed point $x \in X$ for which the specialization of ω is non-zero in $\mathrm{Br}(x)$. We will in fact prove the following stronger result on the ramification of Brauer groups:

THEOREM 1.1.4 (Corollary 3.2.3). – *Let* $\mathscr{U}$ *be either* $\mathscr{X}$ *itself or its Henselization at a closed point. Put* $U := \mathscr{U}[p^{-1}]$ *and assume that the purity of Brauer groups holds for* $\mathscr{U}$*. If* $\mathscr{X}$ *is Henselian local, then assume further that all irreducible components of the divisor on* $\mathscr{U}$ *defined by the radical of* (p) *are regular. Then the kernel of the map*

$$\psi_x : \mathrm{Br}(U) \longrightarrow \prod_{v \in U_0} \mathbb{Q}/\mathbb{Z}\,, \quad \omega \mapsto (\mathrm{inv}_v(\omega|_v))_{v \in U_0}$$

agrees with $\mathrm{Br}(\mathscr{U})$*, where* U_0 *denotes the set of closed points on* U.

The prime-to-p part of Theorem 1.1.4 has been proved in [10]. We will prove the p-primary part of this result using Kerz's idèle class group [24]. Our method of the proof gives also an alternative proof of the prime-to-part in [10].

1.2. – As an application of Theorem 1.1.3 (2), we give an explicit calculation of $\mathrm{A}_0(X)$ for a potentially rational suface X/k, a proper smooth geometrically connected surface X over k such that $X \otimes_k k'$ is rational for some finite extension k'/k. For such a surface X, the map (1.1.2) has been known to be injective (see Proposition 4.1.2 below), and hence bijective by Theorem 1.1.3 (2). On the other hand, for such a surface X, we have

$$\mathrm{Br}(X)/\mathrm{Br}(k) \simeq H^1_{\mathrm{Gal}}(G_k, \mathrm{NS}(\overline{X})),$$

where $\mathrm{NS}(\overline{X})$ denotes the Néron-Severi group of $\overline{X} := X \otimes_k \overline{k}$, and G_k denotes the absolute Galois group of k. Thus knowing the G_k-module structure of $\mathrm{NS}(\overline{X})$, we can compute $\mathrm{A}_0(X)$ by determining which element of $\mathrm{Br}(X)$ are unramified along the closed fiber of $\mathscr{X}$. For example, consider a cubic surface for $a \in k^\times$

$$X : T_0^3 + T_1^3 + T_2^3 + aT_3^3 = 0 \quad \text{in} \quad \mathbb{P}_k^3 = \mathrm{Proj}(k[T_0, T_1, T_2, T_3]).$$

If a is a cube in k, then X is isomorphic to the blow-up of $\mathbb{P}_k^2$ at six k-valued points in the general position (Shafarevich) and we have $\mathrm{A}_0(X) = 0$. We will prove the following result, which is an extention of results in [10] Example 2.8.

THEOREM 1.2.1 (Theorem 4.1.1). – *Assume that* $\mathrm{ord}_k(a) \equiv 1 \mod (3)$ *and that* k *contains a primitive cubic root of unity. Then we have*

$$\mathrm{A}_0(X) \simeq (\mathbb{Z}/3)^2.$$

In his paper [11], Dalawat provided a method to compute $\mathrm{A}_0(X)$ for a potentially rational surface X, which works under the assumption that the action of G_k on $\mathrm{NS}(\overline{X})$ is unramified. Theorem 1.1.3 provides a new method to compute $\mathrm{A}_0(X)$, which does not require Dalawat's assumption. Note that p may be 3 in Theorem 1.2.1, so that the action of G_k on $\mathrm{NS}(\overline{X})$ may ramify even wildly.

1.3. – Let $\mathfrak{o}_k$ be as before, and let $\mathscr{X}$ be a regular scheme which is proper flat of finite type over $\mathfrak{o}_k$. Assume that $\mathscr{X}$ has *good or semistable reduction* over $\mathfrak{o}_k$. Let d be the absolute dimension of $\mathscr{X}$, and let r be a positive integer. In [35], we proved that the cycle class map

$$\varrho_m^{d-1} : \mathrm{CH}^{d-1}(\mathscr{X})/m \longrightarrow H_{\text{ét}}^{2d-2}(\mathscr{X}, \mu_m^{\otimes d-1})$$

is bijective for any positive integer m prime to p. Here μ_m denotes the étale sheaf of m-th roots of unity. As a new tool to study $\mathrm{CH}^{d-1}(\mathscr{X})$, we introduce the p-adic cycle class map defined in [37] Corollary 6.1.4:

$$\varrho_{p^r}^{d-1} : \mathrm{CH}^{d-1}(\mathscr{X})/p^r \longrightarrow H_{\text{ét}}^{2d-2}(\mathscr{X}, \mathfrak{T}_r(d-1)).$$

Here $\mathfrak{T}_r(n) = \mathfrak{T}_r(n)_{\mathscr{X}}$ denotes the étale Tate twist with $\mathbb{Z}/p^r\mathbb{Z}$-coefficients [37] (see also [38] § 7), which is an object of $D^b(\mathscr{X}, \mathbb{Z}/p^r\mathbb{Z})$, the derived category of bounded complexes of étale $\mathbb{Z}/p^r\mathbb{Z}$-sheaves on $\mathscr{X}$. This object $\mathfrak{T}_r(n)$ plays the role of $\mu_m^{\otimes n}$, and we expect that $\mathfrak{T}_r(n)$ agrees with $\mathbb{Z}(n)^{\text{ét}} \otimes^{\mathbb{L}} \mathbb{Z}/p^r\mathbb{Z}$, where $\mathbb{Z}(n)^{\text{ét}}$ denotes the conjectural étale motivic complex of Beilinson-Lichtenbaum ([2], [26], [37] Conjecture 1.4.1 (1)). Concerning the map $\varrho_{p^r}^{d-1}$, we will prove the following result:

THEOREM 1.3.1. – *The cycle class map* $\varrho_{p^r}^{d-1}$ *is surjective.*

We have nothing to say about the injectivity of $\varrho_{p^r}^{d-1}$ in this paper (compare with [44]). A key to the proof of Theorem 1.3.1 is the non-degeneracy of a canonical pairing of finite $\mathbb{Z}/p^r\mathbb{Z}$-modules

$$H_{\text{ét}}^{2d-2}(\mathscr{X}, \mathfrak{T}_r(d-1)) \times H_{Y,\text{ét}}^3(\mathscr{X}, \mathfrak{T}_r(1)) \longrightarrow \mathbb{Z}/p^r\mathbb{Z}$$

proved in [37] Theorem 10.1.1. We explain an outline of the proof of Theorem 1.3.1. Let Y, U, A_x be as in Theorem 1.1.4. Let X_0 and Y_0 be the sets of all closed points on X and Y, respectively, and let $\mathrm{sp} : X_0 \to Y_0$ be the specialization map of points. By the duality mentioned above, there is an isomorphism of finite groups

$$H_{\text{ét}}^{2d-2}(\mathscr{X}, \mathfrak{T}_r(d-1)) \xrightarrow{\sim} H_{Y,\text{ét}}^3(\mathscr{X}, \mathfrak{T}_r(1))^*,$$

where we put $M^* := \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$ for abelian group M. We will construct an injective map

$$\theta_{p^r} : H^3_{Y,\text{ét}}(\mathscr{X}, \mathfrak{T}_r(1)) \hookrightarrow \prod_{x \in U_0} {}_{p^r}\mathrm{Br}(A_x[p^{-1}])$$

whose dual fits into a commutative diagram

$$\begin{array}{ccccc}
\mathrm{CH}^{d-1}(\mathscr{X})/p^r & \xrightarrow{\varrho^{d-1}_{\ell^r}} & H^{2d-2}_{\text{ét}}(\mathscr{X}, \mathfrak{T}_r(d-1)) & \xrightarrow{\sim} & H^3_{Y,\text{ét}}(\mathscr{X}, \mathfrak{T}_r(1))^* \\
\uparrow & & & & \uparrow (\theta_{p^r})^* \\
\bigoplus_{x \in U_0} \bigoplus_{v \in \mathrm{Spec}(A_x[p^{-1}])_0} \mathbb{Z}/p^r\mathbb{Z} & & \xrightarrow{(\psi_{p^r})^*} & & \bigoplus_{x \in U_0} \left({}_{p^r}\mathrm{Br}(A_x[p^{-1}])\right)^*.
\end{array}$$

Here ψ_{p^r} denotes the direct product of the p^r-torsion part of the map ψ_x in Theorem 1.1.4 for all $x \in U_0$, which is injective by Theorem 1.1.4 and its dual $(\psi_{p^r})^*$ is surjective. Therefore Theorem 1.3.1 will follow from this commutative diagram and the surjectivity of $(\theta_{p^r})^*$ and $(\psi_{p^r})^*$ (see § 6 for details).

1.4. – This paper is organized as follows. In § 3, we will prove Theorem 1.1.4 in a stronger form. In § 4, we compute A_0 of cubic surfaces to prove Theorem 1.2.1. In § 5 and § 6, we will prove Theorem 1.1.3 and Theorem 1.3.1, respectively.

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Notation

1.5. – For an abelian group M and a positive integer n, ${}_nM$ and M/n denote the kernel and the cokernel of the map $M \xrightarrow{\times n} M$, respectively. For a field k, $\overline{k}$ denotes a fixed separable closure, and G_k denotes the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$. For a discrete G_k-module M, $H^*(k, M)$ denotes the Galois cohomology groups $H^*_{\mathrm{Gal}}(G_k, M)$, which are the same as the étale cohomology groups of $\mathrm{Spec}(k)$ with coefficients in the étale sheaf associated with M.

1.6. – Unless indicated otherwise, all cohomology groups of schemes are taken over the étale topology. For a commutative ring R with unity and a sheaf $\mathscr{F}$ on $\mathrm{Spec}(R)_{\text{ét}}$, we often write $H^*(R, \mathscr{F})$ for $H^*(\mathrm{Spec}(R), \mathscr{F})$.