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THE HOROBOUNDARY OF OUTER SPACE, AND GROWTH UNDER RANDOM AUTOMORPHISMS

BY CAMILLE HORBEZ

ABSTRACT. – We show that the horoboundary of outer space for the Lipschitz metric is a quotient of Culler and Morgan's classical boundary, two trees being identified whenever their translation length functions are homothetic in restriction to the set of primitive elements of F_N. We identify the set of Busemann points with the set of trees with dense orbits. We also investigate a few properties of the horoboundary of outer space for the backward Lipschitz metric, and show in particular that it is infinite-dimensional when $N \geq 3$. We then use our description of the horoboundary of outer space to derive an analogue of a theorem of Furstenberg and Kifer [20] and Hennion [32] for random products of outer automorphisms of F_N, that estimates possible growth rates of conjugacy classes of elements of F_N under such products.

RÉSUMÉ. – Nous montrons que l'horofrontière de l'outre-espace pour la distance de Lipschitz est un quotient de la frontière classique de Culler et Morgan, dans laquelle deux arbres sont identifiés lorsque leurs fonctions-longueurs de translation sont homothétiques en restriction aux éléments primitifs de F_N. Nous identifions l'ensemble des points de Busemann à l'ensemble des arbres à orbites denses. Nous étudions également quelques propriétés de l'horofrontière de l'outre-espace pour la distance de Lipschitz inversée, et montrons en particulier que celle-ci est de dimension topologique infinie dès que $N \geq 3$. Nous utilisons ensuite notre description de l'horofrontière de l'outre-espace pour montrer un analogue d'un théorème de Furstenberg et Kifer [20] et Hennion [32] pour les produits aléatoires d'automorphismes extérieurs de F_N, estimant les taux de croissance possibles des classes de conjugaison d'éléments de F_N sous l'action de tels produits.

Introduction

Over the past decades, the study of the group $\mathrm{Out}(F_N)$ of outer automorphisms of a free group of rank N has benefited a lot from the study of its action on some geometric complexes, among which stands Culler and Vogtmann's outer space [12]. A main source of inspiration in this study comes from analogies with arithmetic groups acting on symmetric spaces, and mapping class groups of surfaces acting on Teichmüller spaces. *Outer space* CV_N (or its unprojectivized version cv_N) is the space of equivariant homothety (isometry) classes of simplicial free, minimal, isometric actions of F_N on simplicial metric trees. It is naturally equipped with an asymmetric metric d (i.e., d satisfies the separation axiom and the triangle inequality, but we can have $d(x,y) \neq d(y,x)$). This metric is defined in analogy with Thurston's asymmetric metric on Teich(S). The distance between two trees $T, T' \in CV_N$ is the logarithm of the infimal Lipschitz constant of an F_N-equivariant map from the covolume one representative of T to the covolume one representative of T' [15]. We aim at giving a description of the horoboundary of outer space, which we then use to derive a statement about the growth of elements of F_N under random products of automorphisms, analogous to a theorem of Furstenberg and Kifer [20] and Hennion [32] about random products of matrices.

The horoboundary of a metric space was introduced by Gromov in [22]. Let (X,d) be a metric space, and b be a basepoint in X. Associated to any $z \in X$ is a continuous map

$$\begin{aligned} \psi_z : X &\to \mathbb{R} \\ x &\mapsto d(x,z) - d(b,z). \end{aligned}$$

Let $\mathcal{C}(X)$ be the space of real-valued continuous functions on X, equipped with the topology of uniform convergence on compact sets. Under some geometric assumptions on X, the map

$$\begin{aligned} \psi : X &\to \mathcal{C}(X) \\ z &\mapsto \psi_z \end{aligned}$$

is an embedding, and taking the closure of its image yields a compactification of X, called the *horofunction compactification*. The space $\overline{\psi(X)} \smallsetminus \psi(X)$ is called the *horoboundary* of X. In [62], Walsh extended this notion to the case of asymmetric metric spaces.

Walsh identified the horofunction compactification of the Teichmüller space of a closed surface, with respect to Thurston's asymmetric metric, with Thurston's compactification, defined as follows (see [14]). Let $\mathcal{C}(S)$ denote the set of free homotopy classes of simple closed curves on S. The space Teich(S) embeds into $\mathbb{PR}^{\mathcal{C}(S)}$ by sending any element to the collection of all lengths of geodesic representatives of homotopy classes of simple closed curves, and the image of this embedding has compact closure. Thurston identified the boundary with the space of projectivized measured laminations on S.

In the context of group actions on trees, lengths of curves are replaced by translation lengths of elements of the group. The *translation length* of an element g of a group G acting by isometries on an $\mathbb{R}$-tree T is defined as $||g||_T := \inf_{x \in T} d_T(x, gx)$. Looking at the translation lengths of all elements of F_N yields an embedding of cv_N into $\mathbb{R}^{F_N}$, whose image has projectively compact closure, as was proved by Culler and Morgan [11]. This compactification $\overline{CV_N}$ of outer space was described by Cohen and Lustig [10] and Bestvina and Feighn [6] as the space of homothety classes of minimal, very small, isometric actions of F_N on $\mathbb{R}$-trees, see also [37].

We prove that Culler and Morgan's compactification of outer space is not isomorphic to the horofunction compactification. To get the horocompactification of outer space, one has to restrict translation length functions to the set $\mathcal{P}_N$ of primitive elements of F_N, i.e., those elements that belong to some free basis of F_N. This yields an embedding of CV_N into $\mathbb{PR}^{\mathcal{P}_N}$, whose image has compact closure $\overline{CV_N}^{\mathrm{prim}}$, called the *primitive compactification* [36]. Alternatively, the space $\overline{CV_N}^{\mathrm{prim}}$ is the quotient of $\overline{CV_N}$ obtained by identifying two trees whenever their translation length functions are equal in restriction to $\mathcal{P}_N$. An explicit description of this equivalence relation in terms of trees was given in [36, Theorem 0.2]. The equivalence class of a tree with dense F_N-orbits consists of a single point. The typical example of a nontrivial equivalence class is obtained by equivariantly folding an edge e of the Bass-Serre tree of a splitting of the form $F_N = F_{N-1}*$ along some translate ge, where $g \in F_{N-1}$ is not contained in any proper free factor of F_{N-1}.

THEOREM 2.2. – *There exists a unique* $\mathrm{Out}(F_N)$*-equivariant homeomorphism from* $\overline{CV_N}^{\mathrm{prim}}$ *to the horocompactification of* CV_N *which restricts to the identity on* CV_N*. For all* $z \in \overline{CV_N}^{\mathrm{prim}}$*, the horofunction associated to* z *is given by*

$$\psi_z(x) = \log \sup_{g \in \mathcal{P}_N} \frac{||g||_z}{||g||_x} - \log \sup_{g \in \mathcal{P}_N} \frac{||g||_z}{||g||_b}$$

for all $x \in CV_N$ *(identified with its covolume* 1 *representative).*

Both suprema in the above formula can be taken over a finite set of elements that only depends on x and b. We could also choose any representative of z in $\overline{cv_N}$, and take the supremum over all elements of F_N. Denoting by $\mathrm{Lip}(x, z)$ the infimal Lipschitz constant of an F_N-equivariant map from x to a fixed representative of z in $\overline{cv_N}$, we also have

$$\psi_z(x) = \log \mathrm{Lip}(x, z) - \log \mathrm{Lip}(b, z).$$

A special class of horofunctions in the horoboundary of a metric space X comes from points arising as limits of infinite almost-geodesic rays in X, called *Busemann points* [57]. Walsh proved that all points in the horoboundary of the Teichmüller space of a closed surface are Busemann. This is no longer true in outer space, one obstruction coming from the noncompleteness of outer space, see [2]: some points in the boundary are reached in finite time along geodesic intervals. We show that Busemann points in the horoboundary of outer space coincide with trees having dense orbits under the F_N-action.

As the Lipschitz metric on outer space is not symmetric, one can also consider the horoboundary of outer space for the backward metric. We investigate some of its properties, but we only give a complete description when $N = 2$. There seems to be some kind of duality between the two boundaries we get, the horofunctions for the backward metric being expressed in terms of dual currents. Topologically though, both boundaries are of rather different nature. For example, we show that the backward horocompactification has infinite topological dimension when $N \geq 3$, while the forward horocompactification of outer space has dimension $3N - 4$.

Our motivation for understanding the horoboundary of outer space comes from the question of describing the behavior of random walks on $\mathrm{Out}(F_N)$. Karlsson and Ledrappier proved that a typical trajectory of the random walk on a locally compact group G acting by isometries on a proper metric space X follows a (random) direction, given by a point in the horofunction compactification of X, see [47, 48].

Given a probability measure μ on a group G, the *left random walk* on (G, μ) is the Markov chain on G whose initial distribution is given by the Dirac measure at the identity