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THE OBSTACLE PROBLEM FOR THE TOTAL VARIATION FLOW

BY VERENA BÖGELEIN, FRANK DUZAAR AND CHRISTOPH SCHEVEN

ABSTRACT. – We prove existence results for the obstacle problem related to the total variation flow. For sufficiently regular obstacles the solutions are obtained via the method of minimizing movements. The results for more general obstacles are derived by approximation with regular obstacles in the sense of a stability property of solutions with respect to the obstacle. Finally, we present the treatment of the evolutionary counterpart of a classical stationary result concerning minimal surfaces with thin obstacles by means of the $(n-1)$-dimensional variational measure introduced by De Giorgi, Colombini and Piccinini.

RÉSUMÉ. – Nous démontrons des résultats d'existence pour le problème de l'obstacle lié au flot de variation totale. Pour les obstacles suffisamment réguliers, nous obtenons les solutions via le procédé de minimisation des mouvements. Les résultats pour les obstacles plus généraux sont dérivés par approximation avec des obstacles réguliers dans le sens d'une propriété de stabilité de solutions relative à l'obstacle. Enfin, nous présentons le traitement de la contrepartie parabolique d'un résultat classique concernant les surfaces minimales avec des obstacles minces au moyen de la mesure variationnelle $(n-1)$-dimensionnelle introduite par De Giorgi, Colombini et Piccinini.

1. Introduction and results

The total variation flow

$$\partial_t u - \operatorname{div}\left(\frac{Du}{|Du|}\right) = 0 \tag{1.1}$$

is an important prototype example of a nonlinear parabolic equation. The equation is one of the borderline cases of the parabolic p-Laplacian equation, namely the case $p = 1$, and therefore (1.1) is often called the parabolic 1-harmonic flow. Formally, the equation can be interpreted as the L^2-gradient flow associated to the 1-energy. As it is well known, problems with linear growth find their natural formulation in the framework of functions of bounded variation, for short BV-functions. The precise setup shall be given later in § 1.1.

There is a large interest concerned with this equation, and we refer to [3, 4, 5, 6, 7, 10] for the first generalized (weak) formulations of (1.1); see also the monograph [8]. These concepts rely on the Anzellotti pairing [9] and the existence proofs are based on nonlinear semi-group theory, in particular on techniques of completely accretive operators and Crandall & Liggett's semigroup generation theorem. Another approach, introduced by Lichnewsky & Temam in [32], suggests the interpretation of (1.1) in terms of the generating 1-energy in the sense that solutions of the associated Dirichlet problem solve a variational inequality. Roughly speaking and on a purely formal level, a solution $u = u(x,t)$ to the Dirichlet problem associated to (1.1) on a space-time cylinder $\Omega_T = \Omega \times (0,T)$ (Ω a bounded domain in $\mathbb{R}^n$ and $T > 0$) for given initial values $u_o \colon \Omega \to \mathbb{R}$, can be interpreted as a solution of the variational inequality

$$\begin{aligned} \iint_{\Omega_T} |Du| \mathrm{d}x\mathrm{d}t \le \iint_{\Omega_T} \big[\partial_t v(v-u) + |Dv|\big] \mathrm{d}x\mathrm{d}t \\ - \tfrac{1}{2}\|(v-u)(T)\|^2_{L^2(\Omega)} + \tfrac{1}{2}\|v(0) - u_o\|^2_{L^2(\Omega)} \end{aligned} \tag{1.2}$$

for any (sufficiently regular) comparison function $v \colon \Omega_T \to \mathbb{R}$ coinciding with u on the lateral boundary $\partial\Omega \times (0,T)$. According to [32] solutions of the variational inequality are termed *pseudo solutions* or *variational solutions*. The viewpoint of pseudo solutions to variational inequalities has been adopted in [13] for the treatment of gradient flows related to functionals arising in image restoration problems, for example the famous Rudin, Osher & Fatemi image denoising model [36]; see also [12] for flows related to convex, coercive variational integrands.

In this work, we are concerned with the so-called *obstacle problem* related to the total variation flow equation, meaning that we are interested in solutions of (1.1) subject to the additional pointwise *obstacle constraint* that they lie above a given obstacle function $\psi \colon \Omega_T \to \mathbb{R}$. For the obstacle function at the initial time one poses the compatibility condition $\psi(0) := \psi(\cdot, 0) \ge u_o$ on Ω. This leads to the variational inequality (1.2) in the sense that a function u solves the obstacle problem to the total variation flow if $u \ge \psi$ on Ω_T and the variational inequality (1.2) holds true for any comparison map v with the same boundary values as u on the lateral boundary and such that $v \ge \psi$; see Definition 1.1 below for the precise notion of solution. Classic references for the obstacle problem related to the parabolic p-Laplacian, respectively the porous medium equation are [1], the monograph [34], and more recent ones [14, 15, 16, 37]. An alternative approach to obstacle problems would be the construction of the smallest supersolution to the total variation flow equation staying above the obstacle function ψ. This point of view, which plays a fundamental role in any nonlinear potential theory, is applied for parabolic p-Laplacian (type) equations in [31, 33] and more recently for the porous medium equation in [30].

Our main concern in this paper is to build up a satisfactory existence theory for the obstacle problem for the total variation flow. The challenge here is to find the proper formulation of the obstacle problem, making possible a sufficiently general existence theory, which, for example, allows the treatment of obstacle functions modeling thin obstacles. Such a theory could also be one of the building blocks for the definition of a parabolic 1-capacity.

1.1. Formulation of the obstacle problem

The rigorous formulation takes place in the parabolic function space $L^p_{w*}(0,T; \mathrm{BV}(\Omega))$, consisting of those maps $v \colon (0,T) \to \mathrm{BV}(\Omega)$ which are weakly*-measurable and such that $t \mapsto \|Dv(t)\|(\Omega)$ is in $L^p(0,T)$; see § 2 for the precise definition and the notion of the total variation $\|Dv(t)\|(\Omega)$. As it is well known, dealing with boundary values for functions of bounded variation is a delicate issue, since the trace operator is not continuous with respect to the weak* convergence in $\mathrm{BV}(\Omega)$. To overcome this problem, we consider a slightly larger reference domain Ω^* compactly containing the bounded open set Ω. Then, given a reference function $u_o \in \mathrm{BV}(\Omega^*)$, the Dirichlet boundary condition $u = u_o$ on $\partial\Omega$ for a function $u \in \mathrm{BV}(\Omega^*)$ is defined by requiring that $u = u_o$ a.e. on $\Omega^* \setminus \overline{\Omega}$. For functions with this property we write $u \in \mathrm{BV}_{u_o}(\Omega)$ for short. The space $L^p_{w*}(0,T;\mathrm{BV}_{u_o}(\Omega))$ is defined as the space of functions $u \in L^p_{w*}(0,T;\mathrm{BV}(\Omega^*))$ such that for almost all time slices t the map $u(t) := u(\cdot,t)$ belongs to $\mathrm{BV}_{u_o}(\Omega)$. In terms of the described notion of boundary values the obstacle problem for the total variation flow can be formulated as follows. We consider initial data $u_o\colon \Omega^* \to \mathbb{R}$ with

$$u_o \in L^2(\Omega^*) \cap \mathrm{BV}(\Omega^*), \tag{1.3}$$

and obstacle functions $\psi\colon \Omega^*_T \to \mathbb{R}$ with

$$\psi \in L^2\big(\Omega^*_T\big) \cap L^1_{w*}\big(0,T;\mathrm{BV}_{u_o}(\Omega)\big). \tag{1.4}$$

Moreover, we postulate that ψ admits initial values $\psi(0)$ in the $L^2(\Omega^*)$-sense, satisfying the compatibility condition $\psi(0) \le u_o$ a.e. in Ω^*. Finally, we assume that there exists a sufficiently regular extension g of the initial datum u_o to Ω^*_T, more precisely a mapping $g\colon \Omega^*_T \to \mathbb{R}$ such that

$$\begin{cases} g \in L^1_{w*}(0,T;\mathrm{BV}_{u_o}(\Omega)) \text{ with } \partial_t g \in L^2(\Omega^*_T), \\ \qquad g(0) = u_o \text{ and } g \ge \psi \text{ a.e. on } \Omega_T. \end{cases} \tag{1.5}$$

The following definition gives the notion of *variational solution* to the obstacle problem for the total variation flow, that we will use in this paper. In a certain sense, the concept here seems to be the natural extension of the classical definition of pseudo solutions given by Lichnewsky & Temam in [32].

Definition 1.1 (Variational Solutions). – Assume that the Cauchy-Dirichlet datum u_o and the obstacle ψ fulfill the hypotheses (1.3) and (1.4). Moreover, assume that the compatibility condition (1.5) holds true. We identify a measurable map $u\colon \Omega^*_T \to \mathbb{R}$ in the class

$$u \in L^\infty\big([0,T];L^2(\Omega^*)\big) \cap L^1_{w*}\big(0,T;\mathrm{BV}_{u_o}(\Omega)\big) \text{ with } u \ge \psi \text{ a.e. in } \Omega_T$$

as a *variational solution* to the obstacle problem for the total variation flow if and only if the variational inequality

$$\begin{aligned} \int_0^\tau \|Du\|(\Omega^*)\mathrm{d}t \le \iint_{\Omega^*_\tau} \partial_t v(v-u)\mathrm{d}x\mathrm{d}t + \int_0^\tau \|Dv\|(\Omega^*)\mathrm{d}t \\ - \tfrac12\|(v-u)(\tau)\|^2_{L^2(\Omega^*)} + \tfrac12\|v(0)-u_o\|^2_{L^2(\Omega^*)} \end{aligned} \tag{1.6}$$

holds true, for a.e. $\tau \in [0,T]$ and any $v \in L^1_{w*}(0,T;\mathrm{BV}_{u_o}(\Omega))$ with $\partial_t v \in L^2(\Omega^*_T)$, $v(0) \in L^2(\Omega^*)$ and $v \ge \psi$ a.e. in Ω_T.

Observe, by assumption (1.5) the map g is an admissible comparison function in the variational inequality (1.6). This allows the testing of (1.6) by $v = g$, and leads to certain energy bounds. In particular one can conclude that variational solutions attain the initial