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On the cycle class map for zero-cycles over local fields
ON THE CYCLE CLASS MAP FOR ZERO-CYCLES OVER LOCAL FIELDS

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WITH AN APPENDIX BY SPENCER BLOCH

ABSTRACT. – We study the Chow group of 0-cycles of smooth projective varieties over local and strictly local fields. We prove in particular the injectivity of the cycle class map to integral $\ell$-adic cohomology for a large class of surfaces with positive geometric genus, over local fields of residue characteristic $\neq \ell$. The same statement holds for semistable $K3$ surfaces defined over $\mathbb{C}((t))$, but does not hold in general for surfaces over strictly local fields.

RéSUMÉ. – Nous étudions le groupe de Chow des 0-cycles des variétés projectives et lisses sur les corps locaux et strictement locaux. Nous prouvons en particulier l’injectivité de l’application classe de cycle vers la cohomologie $\ell$-adique entière pour de nombreuses surfaces de genre géométrique non nul, sur les corps locaux de caractéristique résiduelle $\neq \ell$. Le même énoncé vaut pour les surfaces $K3$ semi-stables définies sur $\mathbb{C}((t))$, mais ne vaut pas en général pour les surfaces sur les corps strictement locaux.

1. Introduction

Let $X$ be a smooth projective variety over a field $K$, let $\text{CH}_0(X)$ denote the Chow group of 0-cycles on $X$ up to rational equivalence and let $A_0(X) \subset \text{CH}_0(X)$ be the subgroup of cycle classes of degree 0.

When $K$ is algebraically closed, the group $A_0(X)$ is divisible and its structure as an abelian group is, conjecturally, rather well understood, thanks to Roitman’s theorem and to the Bloch-Beilinson-Murre conjectures. A central tool for the study of $A_0(X)$ over other types of fields is the cycle class map

$$\psi : \text{CH}_0(X)/n\text{CH}_0(X) \to H^{2d}_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}(d))$$

to étale cohomology, where $n$ denotes an integer invertible in $K$ and $d = \dim(X)$. The group $H^{2d}_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}(d))$ is easier to understand, thanks to the Hochschild-Serre spectral sequence.

The first author is supported by the the ERC Advanced Grant 226257, the Chaire d’excellence 2011 of the Fondation Sciences Mathématiques de Paris and the Einstein Foundation.
sequence; for instance, if $K$ has cohomological dimension $\leq 1$ and $X$ is simply connected, then $H^d_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}(d)) = \mathbb{Z}/n\mathbb{Z}$ and $\psi$ may be interpreted as the degree map.

According to one of the main results of higher-dimensional unramified class field theory, due to Kato and Saito [43], if $K$ is a finite field, the group $A_0(X)$ is finite and $\psi$ is an isomorphism. More recently, Saito and Sato [69] have shown that if $K$ is the quotient field of an excellent Henselian discrete valuation ring with finite or separably closed residue field, the group $A_0(X)$ is the direct sum of a finite group of order prime to $p$ and a group divisible by all integers prime to $p$ (see also [11, Théorème 3.25]). In this case, however, the map $\psi$ need not be either injective or surjective. What Saito and Sato prove, instead, is the bijectivity of the analogous cycle class map for cycles of dimension 1 on regular models of $X$ over the ring of integers of $K$ (see [69, Theorem 1.16]).

Following a method initiated by Bloch [4], one may approach the torsion subgroup of $A_0(X)$, as well as the kernel of $\psi$, with the help of algebraic $K$-theory, when $X$ is a surface. We refer to [10] for a detailed account of this cycle of ideas. Strong results were obtained in this way for rational surfaces, and more generally for surfaces with geometric genus zero, over number fields, when $X$ is a surface whose Albanese variety has potentially good reduction. If the irreducible components of $A$ satisfy the Tate conjecture, then for any $\ell$ invertible in $k$, the cycle class map

$$CH_0(X) \otimes \mathbb{Z}_\ell \to H^2_{\text{ét}}(X, \mathbb{Z}_\ell(2))$$

is injective. Equivalently, the natural pairing $CH_0(X) \times Br(X) \to \mathbb{Q}/\mathbb{Z}$ is non-degenerate on the left modulo the maximal $\ell$-divisible subgroup of $CH_0(X)$.

The assumption on the irreducible components of $A$ holds as soon as $X$ has geometric genus zero, as well as in many examples of nontrivial degenerations of surfaces with nonzero geometric genus (see § 3). Theorem 1 is due to Saito [68] when $X$ is a surface with geometric genus zero over a $p$-adic field. An example of Parimala and Suresh [62] shows that the assumption on the Albanese variety cannot be removed. Finally, we note that Theorem 1 may be viewed as a higher-dimensional generalization of Lichtenbaum-Tate duality for curves, according to which the natural pairing $CH_0(X) \times Br(X) \to \mathbb{Q}/\mathbb{Z}$ is non-degenerate if $X$ is a smooth proper curve over a $p$-adic field (see [54]).
Our starting point for the proof of Theorem 1 is the theorem of Saito and Sato alluded to above about the cycle class map for 1-cycles on $\mathcal{X}$ [69, Theorem 1.16], which allows us to express the kernel of $\psi$ purely in terms of the scheme $A$ and of the cohomology of $X$, when $k$ is either finite or separably closed (Theorem 2.1 and Theorem 2.2). The dimension of $X$ plays no role in this part of the argument; an application to the study of 0-cycles on a cubic threefold over $\mathbb{Q}_p$ may be found in Example 2.12. Theorem 1 is then obtained by analyzing the various cohomology groups which appear in the resulting expression for $\text{Ker}(\psi)$. More precisely, in the situation of Theorem 1, we prove the stronger assertion that the 1-dimensional cycle class map $\psi_{1,A} : \text{CH}^1(A) \otimes \mathbb{Z}_\ell \to H^4_{\text{ét}}(X, \mathbb{Z}_\ell(2))$ to integral $\ell$-adic étale homology is surjective. This provides a geometric explanation for the assumption that the Albanese variety of $X$ have potentially good reduction, a condition which first appeared in [68] and which turns out to be essential for the surjectivity of $\psi_{1,A}$ to hold (see Lemma 3.7 and §4.3).

When the residue field $k$ is separably closed instead of finite, the arguments used in the proof of Theorem 1 fail in several places. They still lead to the following statement, which may also be deduced from results of Colliot-Thélène and Raskind [13] (see §4 for comments on this point).

**Theorem 2** (Theorem 4.1 and Remark 4.3). – Assume $k$ is separably closed and $K$ has characteristic 0. If $X$ is a surface with geometric genus zero, then for any $\ell$ invertible in $k$, the cycle class map

$$\text{CH}^0(X) \otimes \mathbb{Z}_\ell \to H^3_{\text{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$$

is injective. If in addition $X$ is simply connected, then $A_0(X)$ is divisible by $\ell$ and the unramified cohomology group $H^3_{\text{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ vanishes.

This leaves open the question of the injectivity of the cycle class map (1.2) when $k$ is separably closed and $X$ is a surface with positive geometric genus over $K$. In this situation, the 1-dimensional cycle class map $\psi_{1,A}$ is far from being surjective. Building on the work of Kulikov, Persson, Pinkham [51] [64], and of Miranda and Morrison [59], we nevertheless give a positive answer for semistable $K3$ surfaces over $\mathbb{C}((t))$.

**Theorem 3** (Theorem 5.1). – Let $X$ be a $K3$ surface over $\mathbb{C}((t))$. If $X$ has semistable reduction, the group $A_0(X)$ is divisible.

The proof of Theorem 3 hinges on the precise knowledge of the combinatorial structure of a degeneration of $X$. It would go through over the maximal unramified extension of a $p$-adic field, as far as prime-to-$p$ divisibility is concerned, if similar knowledge were available. This is in marked contrast with the situation over $p$-adic fields, where such knowledge is not necessary for the proof of Theorem 1.

In the final section of this paper, with the help of Ogg-Shafarevich theory and of a construction due to Persson, we show that the hope for a statement analogous to Theorem 1 over the quotient field of a strictly Henselian excellent discrete valuation ring is in fact too optimistic, even over the maximal unramified extension of a $p$-adic field.