ISSN 0012-9593

quatrième série - tome 52 *fascicule 5* *septembre-octobre 2019*

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Bertrand DEROIN & Nicolas THOLOZAN

Supra-maximal representations from fundamental groups of punctured spheres to $\mathrm{PSL}(2,\mathbb{R})$

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 420 euros.
Abonnement avec supplément papier :
Europe : 551 €. Hors Europe : 620 € ($ 930). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Stéphane Seuret
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 52, 2019, p. 1305 à 1329

SUPRA-MAXIMAL REPRESENTATIONS FROM FUNDAMENTAL GROUPS OF PUNCTURED SPHERES TO PSL(2, ℝ)

BY Bertrand DEROIN AND Nicolas THOLOZAN

ABSTRACT. – We study a particular class of representations from the fundamental groups of punctured spheres $\Sigma_{0,n}$ to the group $\mathrm{PSL}(2,\mathbb{R})$, which we call *supra-maximal*. Though most of them are Zariski dense, we show that supra-maximal representations are *totally non hyperbolic*, in the sense that every *simple* closed curve is mapped to an elliptic or parabolic element. They are also shown to be *geometrizable* (apart from the reducible ones) in the following very strong sense : for any element of the Teichmüller space $\mathscr{T}_{0,n}$, there is a unique holomorphic equivariant map with values in the lower half-plane $\mathbb{H}^-$. In the relative character varieties, the components of supra-maximal representations are shown to be compact and symplectomorphic (with respect to the Atiyah-Bott-Goldman symplectic structure) to the complex projective space of dimension $n-3$ equipped with a certain multiple of the Fubini-Study form that we compute explicitly. This generalizes a result of Benedetto-Goldman [3] for the sphere minus four points.

RÉSUMÉ. – Nous étudions une classe particulière de représentations du groupe fondamental des sphères épointées $\Sigma_{0,n}$ dans le groupe $\mathrm{PSL}(2,\mathbb{R})$, que nous appelons *supra-maximales*. Bien qu'elles soient pour la plupart Zariski denses, nous montrons qu'elles sont *totalement non hyperboliques*, au sens où l'image de toute courbe fermée *simple* est elliptique ou parabolique. Nous montrons aussi qu'elles sont *géométrisables* (hormis celles qui sont réductibles) en un sens très fort : pour tout élément de l'espace de Teichmüller $\mathscr{T}_{0,n}$, il existe une unique application équivariante holomorphe à valeurs dans le demi-plan inférieur $\mathbb{H}^-$. Nous montrons également que les représentations supra-maximales forment des composantes compactes des variétés de caractère relatives. Munies de la structure symplectique de Atiyah-Bott-Goldman, ces composantes sont symplectomorphes à l'espace projectif complexe de dimension $n-3$ muni d'un multiple de la forme de Fubini-Study que nous calculons explicitement. Cela généralise un résultat de Benedetto-Goldman pour la sphère à quatre trous.

B.D.'s research was partially supported by ANR-13-BS01-0002.

 doi:10.24033/asens.2410

Introduction

0.1. Overview

Let $\Sigma_{g,n}$ be a surface obtained from a connected oriented closed surface of genus g by removing n points, called the punctures. We assume in the sequel that the Euler characteristic of $\Sigma_{g,n}$ is negative. Throughout the paper, we will denote by $G = \mathrm{PSL}(2,\mathbb{R})$ the group of orientation-preserving isometries of the half-planes $\mathbb{H}^{\pm} = \{z \in \mathbb{C} \mid \pm \mathfrak{im}(z) > 0\}$ equipped with the metrics $\frac{dx^2+dy^2}{y^2}$ of curvature -1, where $z = x + iy$. We denote by $\mathrm{Hom}(\pi_1(\Sigma_{g,n}), G)$ the set of representations from the fundamental group of $\Sigma_{g,n}$ to G, and by $\mathrm{Rep}(\pi_1(\Sigma_{g,n}), G) = \mathrm{Hom}(\pi_1(\Sigma_{g,n}), G)/G$ its quotient by the action of G by conjugation. We will call this latter the *character variety*, even though we do not consider the algebraic quotient (in the sense of geometric invariant theory).

A representation $\rho \in \mathrm{Hom}(\pi_1(\Sigma_{g,n}), G)$ determines a flat oriented $\mathbb{R}\mathbf{P}^1$-bundle over $\Sigma_{g,n}$ which, if we forget the flat connection, is encoded up to isomorphism by a class in $H^2(\Sigma_{g,n}, \mathbb{Z})$, called the Euler class, and denoted $\mathbf{eu}(\rho)$. In the closed case, i.e., when $n = 0$, we have $H^2(\Sigma_{g,0}, \mathbb{Z}) \simeq \mathbb{Z}$, so that the Euler class is an integer that satisfies the well-known Milnor-Wood inequality :

$$|\mathbf{eu}(\rho)| \leq |\chi(\Sigma_{g,0})|, \tag{1}$$

as proved by Wood [28], following an earlier work of Milnor [23]. All the integral values in the interval (1) are achieved on $\mathrm{Hom}(\Sigma_{g,0}, G)$. Goldman proved that the level sets of the Euler class are connected [12], and Hitchin that they are indeed diffeomorphic to vector bundles over some symmetric powers of $\Sigma_{g,0}$ [19]. Goldman also proved in his doctoral dissertation that the Euler class is extremal exactly when the representation is the holonomy of a hyperbolic structure on $\Sigma_{g,0}$ [17]. He conjectured more generally that the components of non-zero Euler class are generically made of holonomies of *branched* $\mathbb{H}^{\pm}$-structures on $\Sigma_{g,0}$ with $k = |\chi(\Sigma_{g,0})| - |\mathbf{eu}|$ branch points (see [16], as well as [26] where the problem is discussed).

This paper is the first in a series aiming at studying the analogous picture on the *relative* character varieties when the surface $\Sigma_{g,n}$ is not closed, namely when $n > 0$. We focus here on a particular family of components of the relative character varieties, that we call *supra-maximal*. They occur only on punctured spheres $\Sigma_{0,n}$ for $n \geq 3$, for some particular choices of elliptic/parabolic peripheral conjugacy classes.

We prove that these components are compact, and more precisely that they are symplectomorphic (with respect to the Atiyah-Bott-Goldman symplectic structure) to the complex projective space of dimension $n - 3$, equipped with a certain multiple of the Fubini-Study form that we compute explicitly. This generalizes to any $n \geq 4$ a result obtained by Benedetto-Goldman in the case $n = 4$ [3].

We also prove that the *supra-maximal representations* (i.e., those lying in supra-maximal components) have very special algebraic and geometric properties. First, we prove that they are totally non hyperbolic, namely that no simple closed curve of $\Sigma_{0,n}$ is mapped to a hyperbolic conjugacy class of G. Moreover, we prove that they are geometrizable by $\mathbb{H}^{-}$-conifolds in a very strong way.

0.2. Volume, relative Euler class and the refined Milnor-Wood inequality

In the closed case, the Euler class is closely related to the *volume* of the representation, classically defined by the integral

$$\mathrm{Vol}(\rho) = \int_{\Sigma_{g,0}} f^*\left(\frac{dx \wedge dy}{y^2}\right) \tag{2}$$

where $f : \widetilde{\Sigma}_{g,0} \to \mathbb{H}^+$ is any ρ-equivariant smooth map. Namely, we have $\mathrm{Vol}(\rho) = -2\pi\, \mathbf{eu}(\rho)$. Burger and Iozzi [5] and Koziarz and Maubon [20] have independently extended the definition of the volume of a representation $\rho : \pi_1(\Sigma_{g,n}) \to G$ to the case of punctured surfaces. (See also Burger-Iozzi-Wienhard [6] for a generalization to representations into Lie groups of Hermitian type.) This volume can be defined as a bounded cohomology class, or more trivially as an integral of the form (2), where the behavior of the equivariant map is constrained in the neighborhood of the cusps: namely, the completion of the metric $f^*\big(\frac{dx^2+dy^2}{y^2}\big)$ in the neighborhood of a cusp is assumed to be a cone, a parabolic cusp, or an annulus with totally geodesic boundary.

The analogous Milnor-Wood inequality

$$|\mathrm{Vol}(\rho)| \le 2\pi|\chi(\Sigma_{g,n})|, \tag{3}$$

holds in this context [6, 20]. It is also proved in [6] that the volume is continuous as a function on $\mathrm{Rep}(\pi_1(\Sigma_{g,n}), G)$ and achieves every value in the interval defined by (3).

The volume heavily depends on the conjugacy class of the peripherals $\rho(c_i)$, where the c_i are elements of $\pi_1(\Sigma_{g,n})$ freely homotopic to positive loops around the punctures. For instance, its reduction modulo 2π equals the sum $-\sum_i R(\rho(c_i))$, where $R(\rho(c_i))$ is the rotation number of $\rho(c_i)$ [6, Theorem 12]. In order to understand better the dependence of the volume on the $\rho(c_i)$, it is convenient to introduce the following function:

$$\theta : G \to \mathbb{R}_+$$

that maps an element $g \in G$ to

- 0 if g is hyperbolic or positive parabolic (i.e., a parabolic that translates the horocycles based at the fixed point of g clockwise),
- 2π if g is negative parabolic (i.e., a parabolic that translates the horocycles based at the fixed point of g counterclockwise) or the identity,
- the value between 0 and 2π of the rotation angle of g when g is elliptic.

We will denote $\theta_i(\rho) = \theta(\rho(c_i))$ and $\Theta(\rho) = \sum_{i=1}^n \theta_i(\rho)$.

Remark 0.1. – The function θ is one among the many ways of lifting the rotation number to a function from G to $\mathbb{R}$. Note however that it is (up to adding a multiple of 2π) the only lift which is continuous in restriction to the set of elliptic elements and upper semi-continuous on the whole group G.

Definition 0.2. – We define the *relative Euler class* of the representation ρ by

$$-\,\mathbf{eu}(\rho) = \frac{1}{2\pi}\big(\mathrm{Vol}(\rho) + \Theta(\rho)\big). \tag{4}$$