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CARLEMAN ESTIMATES FOR GEODESIC X-RAY TRANSFORMS

BY Gabriel P. PATERNAIN AND Mikko SALO

ABSTRACT. – In this article we introduce an approach for studying the geodesic X-ray transform and related geometric inverse problems by using Carleman estimates. The main result states that on compact negatively curved manifolds (resp. nonpositively curved simple or Anosov manifolds), the geodesic vector field satisfies a Carleman estimate with logarithmic weights (resp. linear weights) on the frequency side. As a particular consequence, on negatively curved simple manifolds the geodesic X-ray transform with attenuation given by a general connection and Higgs field is invertible modulo natural obstructions. The proof is based on showing that the Pestov energy identity for the geodesic vector field completely localizes in frequency. Our approach works in all dimensions ≥ 2, on negatively curved manifolds with or without boundary, and for tensor fields of any order.

RÉSUMÉ. – Dans cet article, nous introduisons une approche pour étudier la transformée aux rayons X géodésiques ainsi que des problèmes inverses géométriques qui lui sont rattachés en utilisant des estimations de Carleman. Le résultat principal établit que sur les variétés compactes à courbure strictement négative (resp. les variétés simples à courbure négative ou nulle, ou Anosov), le champ vectoriel géodésique satisfait une estimation de Carleman avec poids logarithmiques (resp. poids linéaires) du côté fréquentiel. En conséquence, modulo des obstructions naturelles, la transformée aux rayons X géodésiques atténuée par une connexion générale et un champ de Higgs est inversible sur les variétés à courbure strictement négative. Un point crucial de la preuve consiste à montrer que l'identité de Pestov pour le champ vectoriel géodésique se localise complètement en fréquence. Notre approche est valide en toutes dimensions ≥ 2, sur les variétés à courbure strictement négative avec ou sans bord, et pour des champs tensoriels de tout ordre.

1. Introduction

Motivation

The geodesic X-ray transform is a central object in geometric inverse problems. In Euclidean space it reduces to the standard X-ray transform, which encodes the integrals of a function over straight lines and provides the mathematical model for imaging methods

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such as X-ray CT and PET [27]. If the underlying medium is not Euclidean, one needs to consider more general curve families. On a Riemannian manifold the geodesic curves provide a natural candidate, and the geodesic X-ray transform encodes the integrals of a function over geodesics. This transform plays an important role (often via linearization or pseudo-linearization arguments) in inverse problems such as

- boundary and lens rigidity on manifolds with boundary [45, 9, 49];
- spectral rigidity on closed manifolds with Anosov geodesic flow [40];
- inverse boundary value problems for hyperbolic equations [50];
- inverse boundary value problems for elliptic equations (Calderón problem) [5, 6].

Several approaches have been introduced for studying the X-ray transform:

- direct methods such as Fourier analysis in symmetric geometries [14];
- microlocal methods based on interpreting the transform or its normal operator as a Fourier integral or pseudodifferential operator [47, 48, 51];
- reductions to PDE and energy methods [46, 39, 11].

In particular, the works [39, 11, 51] involve various L^2 estimates with exponential weights. This suggests that the Carleman estimate methodology, which relies on exponentially weighted L^2 estimates and provides a very powerful general approach to uniqueness results for linear PDE [15, Chapter XXVIII], could have consequences for the geodesic X-ray transform as well. However, as explained in Section 3, the PDE related to the X-ray transform are nonstandard and thus the existing theory of Carleman estimates cannot be directly applied.

This paper presents the first steps toward a Carleman estimate approach for the geodesic X-ray transform. We prove two Carleman estimates for the geodesic vector field, one with logarithmic and another with linear Carleman weights, that have the right form in order to be applied to the geodesic X-ray transform. As a consequence we obtain invertibility results for certain weighted X-ray transforms that correspond to large lower order perturbations in the related PDE, and uniqueness results in related problems such as scattering rigidity and transparent connections. Our approach works in all dimensions ≥ 2, on manifolds with or without boundary, and for tensor fields of any order. However, the main Carleman estimate requires that the manifold has negative curvature, and the Carleman weights are purely on the frequency side (one can think of them as pseudodifferential weights) and thus the estimates do not currently lead to local results for the X-ray transform on the space side as in [51].

Main estimates

Let (M, g) be a compact oriented Riemannian manifold with or without boundary, and let X be the geodesic vector field regarded as a first order differential operator $X : C^\infty(SM) \to C^\infty(SM)$ acting on functions on the unit sphere bundle SM. Our first result is a new energy estimate for X when g is negatively curved. The new estimate involves polynomial weights on the Fourier side and it can be regarded as a *Carleman estimate* for the transport equation $Xu = f$ (see below for analogies with elliptic operators). To describe it we recall some basic harmonic analysis on SM. By considering the vertical Laplacian Δ on each fiber S_xM of SM

we have a natural L^2-decomposition $L^2(SM) = \oplus_{m\geq 0} H_m(SM)$ into vertical spherical harmonics. We set $\Omega_m := H_m(SM) \cap C^\infty(SM)$. Then a function u belongs to Ω_m if and only if $-\Delta u = m(m+d-2)u$ where $d = \dim(M)$ (for details see [13, 4]). If $u \in L^2(SM)$, this decomposition will be written as

$$u = \sum_{m=0}^{\infty} u_m, \qquad u_m \in H_m(SM).$$

We say that u has *finite degree* if the above sum is finite.

Here is our first main result (the norms are $L^2(SM)$ norms).

THEOREM 1.1. – *Let (M, g) be a compact Riemannian manifold with sectional curvature $\leq -\kappa$ where $\kappa > 0$. Let also $\phi_\ell = \log(\ell)$. For any $\tau \geq 1$ and $m \geq 1$, one has*

$$\sum_{\ell=m}^{\infty} e^{2\tau\phi_\ell} \|u_\ell\|^2 \leq \frac{C}{\kappa\tau} \sum_{\ell=m+1}^{\infty} e^{2\tau\phi_\ell} \|(Xu)_\ell\|^2,$$

whenever $u \in C^\infty(SM)$ (with $u|_{\partial(SM)} = 0$ in the boundary case), where C is a positive constant depending only on the dimension of M.

We have a similar estimate for compact manifolds with nonpositive curvature, provided that they are *simple* (simply connected with strictly convex boundary and no conjugate points) or *Anosov* (no boundary, and the geodesic flow satisfies the Anosov property). Both the simple and Anosov conditions can be seen as slightly strengthened forms of the condition that no geodesic has conjugate points (see [40]). However, if the curvature is only nonpositive, the logarithmic weights above need to be replaced by stronger linear weights. We remark that both logarithmic and linear weights have been prominent in the theory of Carleman estimates, see e.g., [16, 19].

THEOREM 1.2. – *Let (M, g) be a simple/Anosov Riemannian manifold having nonpositive sectional curvature. There exist $m_0, \tau_0, \kappa > 0$ such that for any $\tau \geq \tau_0$, for any $m \geq m_0$ and for any $u \in C^\infty(SM)$ of finite degree (with $u|_{\partial(SM)} = 0$ in the boundary case) one has*

$$\sum_{\ell=m}^{\infty} e^{2\tau\phi_\ell} \|u_\ell\|^2 \leq \frac{24}{\kappa e^{2\tau}} \sum_{\ell=m+1}^{\infty} e^{2\tau\phi_\ell} \|(Xu)_\ell\|^2,$$

where $\phi_\ell = \ell$.

Both theorems above are based on weighted frequency localized versions of the *Pestov energy identity*, which has been the main tool in energy methods for X-ray transforms (see e.g., [46, 39, 42]). The Pestov identity has been used in many forms. A powerful recent variant is the inequality proved in [42],

$$\|\nabla_{SM} u\| \leq C \|\overset{\mathrm{v}}{\nabla} X u\|, \tag{1.1}$$

which is valid on any compact simple/Anosov manifold for any $u \in C^\infty(SM)$ (with $u|_{\partial(SM)} = 0$ in the boundary case). Here $\overset{\mathrm{v}}{\nabla}$ is the vertical gradient (see Section 2). So far the Pestov identity has been expressed in terms of L^2 norms, and it has not been known if it is