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PRODUCT AND COPRODUCT IN STRING TOPOLOGY

BY Nancy HINGSTON AND Nathalie WAHL

ABSTRACT. – Let M be a closed Riemannian manifold. We extend the product of Goresky-Hingston, on the cohomology of the free loop space of M relative to the constant loops, to a nonrelative product. It is graded associative and commutative, and compatible with the length filtration on the loop space, like the original product. We prove the following new geometric property of the dual homology coproduct: the nonvanishing of the k-th iterate of the coproduct on a homology class ensures the existence of a loop with a $(k+1)$-fold self-intersection in every representative of the class. For spheres and projective spaces, we show that this is sharp, in the sense that the k-th iterated coproduct vanishes precisely on those classes that have support in the loops with at most k-fold self-intersections. We study the interactions between this cohomology product and the better-known Chas-Sullivan product. We give explicit integral chain level constructions of the loop product and coproduct, including a new construction of the Chas-Sullivan product, which avoids the technicalities of infinite dimensional tubular neighborhoods and delicate intersections of chains in loop spaces.

RÉSUMÉ. – Pour une variété riemannienne M donnée, nous étendons le produit de Goresky-Hingston, sur la cohomologie de l'espace des lacets libres de M relative aux lacets constants, en un produit non relatif. Ce produit est, comme le produit d'origine, associatif, commutatif gradué, et compatible avec la filtration de l'espace des lacets par leur longueur. Nous prouvons la nouvelle propriété géometrique suivante pour le coproduit dual en homologie: la non-trivialité du k-ième itéré du coproduit d'une classe d'homologie implique l'existence d'un lacet s'intersectant lui-même avec multiplicité $(k+1)$ dans toute chaine représentant la classe d'homologie. Pour les sphères et espaces projectifs, on montre que l'implication inverse est aussi vraie: le k-ième itéré du coproduit s'annule précisément sur les classes d'homologie qui ont leur support dans les lacets s'intersectant eux-même avec multiplicité au plus k. Nous étudions les interactions entre ce produit en cohomologie et le produit mieux connu de Chas-Sullivan. Nous donnons une construction explicite des deux produits au niveau des chaines, y compris une nouvelle construction du produit de Chas-Sullivan, qui évite les technicalités des voisinages tubulaires de dimension infinie et les intersections délicates de chaines de lacets.

Let M be a closed oriented manifold of dimension n, for which we pick a Riemannian metric, and let $\Lambda M = \mathrm{Maps}(S^1, M)$ denote its free loop space. Goresky and the

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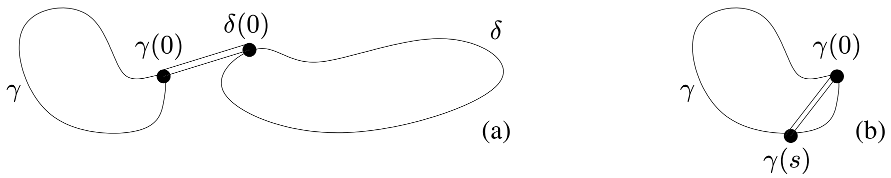

FIGURE 1. The retraction maps R_{CS} and R_{GH}.

first author defined in [16] a product $\circledast$ on the cohomology $H^*(\Lambda M, M)$, relative to the constant loops $M \subset \Lambda M$. They showed that this cohomology product behaves as a form of "dual" of the Chas-Sullivan product [9] on the homology $H_*(\Lambda M)$, at least through the eyes of the energy functional. In the present paper, we lift this relative cohomology product and associated homology coproduct to chain level non-relative product $\widehat{\circledast}: C^*(\Lambda M) \otimes C^*(\Lambda M) \to C^*(\Lambda M)$ and coproduct $\widehat{\vee}: C_*(\Lambda) \to C_*(\Lambda) \otimes C_*(\Lambda)$, on integral singular chains. The operations $\widehat{\circledast}$ and $\widehat{\vee}$ are the "extension by zero" of their relative predecessors $\circledast$ and $\vee$ under the splitting on homology and cohomology induced by the evaluation map $\Lambda M \to M$. We study the algebraic and geometric properties of these two new operations, $\widehat{\vee}$ and $\widehat{\circledast}$, showing in particular that the properties of the original product and coproduct proved in [16], such as associativity, graded commutativity, and compatibility with the length filtration, remain valid for the extended versions. Our main result is the following: for $[A] \in H_*(\Lambda M)$, the non-vanishing of the iterated coproduct $\widehat{\vee}^k[A]$ implies the existence of loops with $(k+1)$-fold intersections in the image of any chain representative of $[A]$; on spheres and projective spaces, we show that this is a complete invariant in the sense that the converse implication also holds.

An advantage of having a coproduct defined on $H_*(\Lambda M)$ is that it makes it possible to study its interplay with the Chas-Sullivan product; we show that the equation $\wedge \circ \vee = 0$ holds. We also exhibit, through computations, the failure of an expected Frobenius equation.

To state our main results more precisely, we need the following ingredients: Fix a small $\varepsilon > 0$ smaller than the injectivity radius and let

$$U_M = \{(x, y) \in M^2 \mid |x - y| < \varepsilon\}$$

be the ε-neighborhood of the diagonal ΔM in M^2, with

$$\tau_M \in C^n(M^2, M^2 \backslash U_M)$$

a Thom class for an associated tubular embedding $\nu_M : TM \xrightarrow{\simeq} U_M$. (A specific model is given in Section 1.3.) Consider the evaluation maps

$$e \times e : \Lambda M^2 \longrightarrow M^2 \quad \text{and} \quad e_I : \Lambda M \times I \longrightarrow M^2$$

taking a pair (γ, δ) to $(\gamma(0), \delta(0))$ and (γ, s) to $(\gamma(0), \gamma(s))$. The tubular embedding ν_M gives rise to a retraction $U_M \to \Delta M$, which we lift to retraction maps

$$\begin{aligned} R_{\mathrm{CS}}: \quad (e \times e)^{-1}(U_M) &\longrightarrow (e \times e)^{-1}(\Delta M) = \{(\gamma, \delta) \in \Lambda M^2 \mid \gamma(0) = \delta(0)\} \\ R_{\mathrm{GH}}: \quad e_I^{-1}(U_M) &\longrightarrow e_I^{-1}(\Delta M) = \{(\gamma, s) \in \Lambda M \times I \mid \gamma(s) = \gamma(0)\} \end{aligned}$$

as depicted in Figure 1: each map adds two geodesic sticks to the loops to make them intersect as required—see Sections 1.4 and 1.5 for precise definitions. Finally, let

$$\text{concat}: \Lambda M \times_M \Lambda M = (e \times e)^{-1}(\Delta M) \longrightarrow M \quad \text{and} \quad \text{cut}: e_I^{-1}(\Delta M) \longrightarrow \Lambda M \times \Lambda M$$

be the concatenation and cutting maps, the latter cutting the loop at its self-intersection time s.

Our results are rooted in the following chain-level construction of the Chas-Sullivan product and the coproduct, based on the ideas of Cohen and Jones in [11] but avoiding the technicalities of infinite dimensional tubular neighborhoods, and avoiding the subtle limit arguments of [16].

THEOREM A. – *The Chas-Sullivan* $\wedge$ *product of* [9] *and the coproduct* $\vee$ *of Goresky-Hingston* [16] *(see also Sullivan* [32]*) admit the following integral chain level descriptions: for* $A \in C_p(\Lambda M)$ *and* $B \in C_q(\Lambda M)$,

$$A \wedge B = (-1)^{n-np} \operatorname{concat}\left(R_{\mathrm{CS}}((e \times e)^*(\tau_M) \cap (A \times B))\right),$$

and for $C \in C_k(\Lambda M, M)$,

$$\vee C = \operatorname{sgn}(\operatorname{cut}(R_{\mathrm{GH}}(e_I^*(\tau_M) \cap (C \times I))))$$

with $A \wedge B \in C_{p+q-n}(\Lambda)$ *and* $\vee C \in \bigoplus_{p+q=k+1-n} C_p(\Lambda M, M) \otimes C_q(\Lambda M, M)$*, with sign change* sgn *given by* $(-1)^{n-np}$ *on the terms of bidegree* (p,q).

We note that the chain complex $C_*(\Lambda M, M) \otimes C_*(\Lambda M, M)$ computes the relative homology group $H_*(\Lambda \times \Lambda, M \times \Lambda \cup \Lambda \times M)$, see Remark 1.5 for more details about this.

In the case of the Chas-Sullivan product, there exist many approaches to chain level constructions, in particular using more "geometric" chains, applying more directly the original idea of Chas and Sullivan of intersecting chains (see e.g., [10, 25, 20]). The idea of using small geodesics to make up for "almost intersections" was already suggested in [33] and can also be found in [13].

We emphasize the similarity between the definitions of $\wedge$ and $\vee$ given above. Note that cutting and concatenating are essentially inverse maps. The product and coproduct are corrected by a sign for better algebraic properties; see Theorems 2.5 and 2.14 and Appendix B for more details about this. In Propositions 3.1 and 3.7, we show that for cycles parametrized by manifolds that, after applying the evaluation map, intersect the diagonal in $M \times M$ transversally, the product and coproduct can be computed by a geometric intersection followed by concatenation or cut maps.

The splitting of $H_*(\Lambda M) \cong H_*(\Lambda M, M) \oplus H_*(M)$ by the evaluation map makes it possible to define an "extension by 0" of the coproduct, setting it to be trivial on the constant loops. The same holds for chains:

THEOREM B. – *There is a unique lift*

$$\widehat{\vee}\colon C_*(\Lambda M) \longrightarrow \bigoplus_{p+q=*+1-n} C_p(\Lambda M) \otimes C_q(\Lambda M)$$

of the coproduct $\vee$ *of Theorem A satisfying that*

$$\langle x \times y, \widehat{\vee} Z\rangle = 0 \quad \textit{if } x \in e^*C^*(M),\ y \in e^*C^*(M),\ \textit{or } Z \in C_*(M)$$

for $e\colon \Lambda M \to M$ *the evaluation at* 0*, and where we identify* M *with the constant loops in* ΛM*. For* $A \in C_*(\Lambda M)$*, with* $p_*A \in C_*(\Lambda M, M)$ *its projection, it is defined by*

$$\widehat{\vee} A = (1 - e_*) \times (1 - e_*) \vee (p_* A).$$