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GENERALIZED SPRINGER CORRESPONDENCE FOR $\mathbf{Z}/m$-GRADED LIE ALGEBRAS

BY WILLE LIU

ABSTRACT. – Let G be a simple simply connected complex algebraic group and let $\mathfrak{g}_*$ be a $\mathbf{Z}/m$-grading on its Lie algebra $\mathfrak{g}$. In a recent series of articles, G. Lusztig and Z. Yun, studied the classification of simple G_0-equivariant perverse sheaves on the nilpotent cone of $\mathfrak{g}_i$ for $i \in \mathbf{Z}/m$, where G_0 is the exponentiation of the degree zero piece $\mathfrak{g}_0$. They proved a decomposition of the equivariant derived category of ℓ-adic sheaves on the nilpotent cone of $\mathfrak{g}_i$ into blocks, each generated by a certain cuspidal local system via *spiral inductions*. We prove a conjecture of them, which predicts the bijectivity of a map from 1) the set of simple perverse sheaves in a fixed block to 2) the set of simple modules of a block of a (trigonometric) degenerate double affine Hecke algebra (dDAHA). This is a dDAHA analogue of the Deligne-Langlands correspondence for affine Hecke algebras proven by Kazhdan-Lusztig. Our results generalize a previous work of E. Vasserot, where the perverse sheaves in the principal block were considered.

RÉSUMÉ. – Soient G un groupe algébrique complexe simple simplement connexe et $\mathfrak{g}_*$ une $\mathbf{Z}/m$-graduation sur $\mathfrak{g} = \operatorname{Lie} G$. Dans une série récente d'articles, G. Lusztig et Z. Yun ont étudié la classification des faisceaux pervers simples G_0-équivariants sur le cône nilpotent de $\mathfrak{g}_i$ pour $i \in \mathbf{Z}/m$, où G_0 est l'exponentialisé de la composante de degré nul $\mathfrak{g}_0$. Ils ont établi une décomposition en blocs de la catégorie dérivée équivariante des faisceaux ℓ-adiques sur le cône nilpotent de $\mathfrak{g}_i$; chacun des blocs est engendré par un certain système local cuspidal via les *inductions spirales*. Nous démontrons leur conjecture qui prédit la bijectivité d'une application de 1) l'ensemble des faisceaux pervers simples dans un bloc donné dans 2) l'ensemble des modules simples d'une algèbre de Hecke doublement affine dégénérée. Ceci est pour les algèbres de Hecke doublement affines dégénérées un résultat analogue à la correspondance de Deligne-Langlands, démontrée par Kazhdan-Lusztig et portant sur les algèbres de Hecke affines. Nos résultats généralisent ceux d'un travail d'E. Vasserot, dans lequel seuls les faisceaux pervers dans le bloc principal étaient pris en compte.

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Introduction

In the present article, we establish a generalized Springer correspondence for $\mathbf{Z}/m$-graded Lie algebras and (trigonometric) degenerate double affine Hecke algebras, which was conjectured by Lusztig-Yun [19]. The main result is Theorem 9.12, which confirms the multiplicity-one conjecture proposed in [19] and can be viewed as a generalized Springer correspondence in the sense of Lusztig [10, 11], for certain degenerate double affine Hecke algebras (dDAHAs) with possibly unequal parameters.

Generalized Springer correspondence

G. Lusztig in [10] generalized the classical Springer correspondence on reductive groups by introducing cuspidal local systems.

Let G be a complex connected reductive group and let $\mathfrak{g} = \operatorname{Lie} G$ denote its Lie algebra, on which G acts by the adjoint action. A *cuspidal pair* $(\mathrm{O}, \mathscr{C})$ on $\mathfrak{g}$ consists of a nilpotent G-orbit $\mathrm{O} \subset \mathfrak{g}^{\mathrm{nil}}$ together with an irreducible G-equivariant local system $\mathscr{C}$ on O such that the following condition holds: for every strict parabolic subalgebra $\mathfrak{p} \subsetneq \mathfrak{g}$ with unipotent radical $\mathfrak{u} \subseteq \mathfrak{p}$ and for any element $x \in \mathrm{O}$, we have $\mathrm{H}^*_c((x + \mathfrak{u}) \cap \mathrm{O}, \mathscr{C}) = 0$.

Given the reductive group G, the following statements are proven in [10]:

1. There is a partition of the set $\operatorname{Irr}\operatorname{Perv}_G(\mathfrak{g}^{\mathrm{nil}})$ of isomorphism classes of simple G-equivariant perverse sheaves on the nilpotent cone $\mathfrak{g}^{\mathrm{nil}}$ into *series*:
$$\operatorname{Irr}\operatorname{Perv}_G(\mathfrak{g}^{\mathrm{nil}}) = \bigsqcup_{\xi} \operatorname{Irr}\operatorname{Perv}_G(\mathfrak{g}^{\mathrm{nil}})_{\xi},$$
where ξ runs over all triples $(M, \mathrm{O}, \mathscr{C})$ up to conjugation, where $M \subset G$ is a Levi subgroup and $(\mathrm{O}, \mathscr{C})$ is a cuspidal pair on the Lie algebra $\mathfrak{m}$ of M.
2. For each triple $\xi = (M, \mathrm{O}, \mathscr{C})$, there is a crystallographic finite Coxeter group W_ξ, called *relative Weyl group*, with a bijection
$$\operatorname{Irr}\operatorname{Perv}_G(\mathfrak{g}^{\mathrm{nil}})_{\xi} \xrightarrow{\sim} \operatorname{Irr}(\mathbf{C}W_\xi\text{-mod}).$$

The series $\operatorname{Irr}\operatorname{Perv}_G(\mathfrak{g}^{\mathrm{nil}})_{\xi}$ is defined using the *Lusztig-Spaltenstein parabolic induction*. Given such a triple $\xi = (M, \mathrm{O}, \mathscr{C})$, let P be a parabolic subgroup of G containing M as Levi factor and let U be the unipotent radical of P with Lie algebra $\mathfrak{u} = \operatorname{Lie} U$. Consider the following diagram of stacks:
$$[G\backslash\mathfrak{g}] \xleftarrow{a} [G\backslash(G \times^P (\mathrm{O} \oplus \mathfrak{u}))] \xrightarrow{\cong} [P\backslash(\mathrm{O} \oplus \mathfrak{u})] \xrightarrow{b} [M\backslash\mathfrak{m}],$$
where $\mathrm{O} \oplus \mathfrak{u}$ is the image of the addition map $\mathrm{O} \times \mathfrak{u} \xrightarrow{+} \mathfrak{p}$.

The complex $\mathbf{I}^\xi = a_* b^* \mathscr{C}[\dim G \times^P (\mathrm{O} \oplus \mathfrak{u})]$ is a G-equivariant semisimple perverse sheaf and $\operatorname{Irr}\operatorname{Perv}_G\left(\mathfrak{g}^{\mathrm{nil}}\right)_\xi$ consists of the simple constituents of $\mathbf{I}^\xi$. The bijection in statement (ii) was constructed by means of a ring isomorphism $\mathbf{C}W_\xi \xrightarrow{\cong} \operatorname{End}(\mathbf{I}^\xi)$.

Equivariant enhancements

Let $\mathrm{D}^{\mathrm{b}}_{G}(\mathfrak{g}^{\mathrm{nil}})$ be the equivariant category of Bernstein-Lunts [1]. In the above statements (i) and (ii), the G-equivariance is only a condition on perverse sheaves and higher extensions between perverse sheaves in $\mathrm{D}^{\mathrm{b}}_{G}(\mathfrak{g}^{\mathrm{nil}})$ play no rôle there.

In [22], higher extensions are taken into account for the statement (i). It is enhanced into the following:

(i$^+$) There is an orthogonal decomposition of triangulated category:

$$\mathrm{D}^{\mathrm{b}}_{G}(\mathfrak{g}^{\mathrm{nil}}) = \bigoplus_{\xi} \mathrm{D}^{\mathrm{b}}_{G}(\mathfrak{g}^{\mathrm{nil}})_{\xi},$$

where $\mathrm{D}^{\mathrm{b}}_{G}(\mathfrak{g}^{\mathrm{nil}})_{\xi}$ is the thick triangulated subcategory generated by $\mathbf{I}^{\xi}$.

For each triple ξ as above, there is a *graded affine Hecke algebra* (graded AHA) $\underline{\mathbf{H}}_{\xi}$: it is a degenerate version of the affine Hecke algebras introduced by G. Lusztig [12], see § 1.1 for the definition of the graded AHA in our setting. Let $\mathbf{C}^{\times}_{q} = \mathbf{C}^{\times}$ be a one-dimensional torus which acts linearly on the Lie algebra $\mathfrak{g}$ by weight -2 and trivially on the group G. The perverse sheaf $\mathbf{I}^{\xi}$ acquires a $G \times \mathbf{C}^{\times}_{q}$-equivariant structure. In [3] and [15], statement (ii) above is enhanced into the following:

(ii$^+$) There is an isomorphism of graded rings

$$\underline{\mathbf{H}}_{\xi} \xrightarrow{\cong} \bigoplus_{k \in \mathbf{Z}} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{G \times \mathbf{C}^{\times}_{q}}(\mathfrak{g}^{\mathrm{nil}})}(\mathbf{I}^{\xi}, \mathbf{I}^{\xi}[k]). \tag{0.1}$$

The summands of the right-hand side vanish except for $k \in 2\mathbf{Z}_{\geq 0}$. Taking quotient by the graded radicals of both sides of (0.1), we recover the aforementioned isomorphism $\mathbf{C}W_{\xi} \cong \mathrm{End}(\mathbf{I}^{\xi})$. One can also replace G with $G \times \mathbf{C}^{\times}_{q}$ in statement (i$^+$).

Perverse sheaves on Z-graded Lie algebras

Keep the reductive group G as above. If $\lambda \in \mathbf{X}_*(G)$ is a cocharacter, then it gives rise to a $\mathbf{Z}$-grading on $\mathfrak{g}$ by

$$\mathfrak{g} = \bigoplus_{n \in \mathbf{Z}} \mathfrak{g}_n, \quad \mathfrak{g}_n = \{x \in \mathfrak{g};\ \lambda(t)x = t^n x, \quad \forall t \in \mathbf{C}^{\times}\}.$$

Fix $\eta \in \mathbf{Z}_{\neq 0}$. The fixed-point subgroup G^{λ} acts on the subspace $\mathfrak{g}_{\eta}$ by the adjoint action. The number of G^{λ}-orbits in $\mathfrak{g}_{\eta}$ is finite. The classification of G^{λ}-equivariant simple perverse sheaves on $\mathfrak{g}_{\eta}$ proven in [11, 15, 16, 3] is the following:

(i$'$) There is a partition of the set $\mathrm{Irr}\,\mathrm{Perv}_{G^{\lambda}}(\mathfrak{g}_{\eta})$ of isomorphism classes of simple G^{λ}-equivariant perverse sheaves on the subspace $\mathfrak{g}^{\lambda}_{\eta}$ into *series*:

$$\mathrm{Irr}\,\mathrm{Perv}_{G^{\lambda}}(\mathfrak{g}_{\eta}) = \bigsqcup_{\xi} \mathrm{Irr}\,\mathrm{Perv}_{G^{\lambda}}(\mathfrak{g}_{\eta})_{\xi},$$

where ξ runs over all triples $(M, \mathrm{O}, \mathscr{C})$ up to G^{λ}-conjugation, where $M \subset G$ is a λ-stable Levi subgroup and $(\mathrm{O}, \mathscr{C})$ is a cuspidal pair on $\mathfrak{m}$ such that $\mathrm{O} \cap \mathfrak{g}_{\eta} \neq \emptyset$.