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THE DEGENERATE HEISENBERG CATEGORY AND ITS GROTHENDIECK RING

BY JONATHAN BRUNDAN, ALISTAIR SAVAGE AND BEN WEBSTER

ABSTRACT. – The degenerate Heisenberg category $\mathcal{H}eis_k$ is a strict monoidal category which was originally introduced in the special case $k = -1$ by Khovanov in 2010. Khovanov conjectured that the Grothendieck ring of the additive Karoubi envelope of his category is isomorphic to a certain $\mathbb{Z}$-form for the universal enveloping algebra of the infinite-dimensional Heisenberg Lie algebra specialized at central charge -1. We prove this conjecture and extend it to arbitrary central charge $k \in \mathbb{Z}$. We also explain how to categorify the comultiplication (generically).

RÉSUMÉ. – La catégorie de Heisenberg dégénérée $\mathcal{H}eis_k$ est une catégorie monoïdale stricte, introduite dans le cas spécial $k = -1$ par Khovanov en 2010. Khovanov a conjecturé que l'anneau de Grothendieck de l'enveloppe additive de Karoubi de sa catégorie est isomorphe à une certaine forme intégrale de l'algèbre enveloppante de l'algèbre de Lie de Heisenberg de dimension infinie, spécialisée à la charge centrale -1. Nous prouvons cette conjecture et l'étendons à une charge centrale arbitraire $k \in \mathbb{Z}$. Nous expliquons également comment catégorifier le coproduit (génériquement).

1. Introduction

Throughout the article, we work over a fixed ground field $\Bbbk$ of characteristic zero. The degenerate Heisenberg category $\mathcal{H}eis_k$ of central charge $k \in \mathbb{Z}$ is a strict $\Bbbk$-linear monoidal category which was introduced originally by Khovanov [9] in the special case $k = -1$, motivated by the calculus of induction and restriction functors between representations of the symmetric groups. Khovanov's definition was extended to arbitrary central charge in [14, 2]. The relations of this category are modeled on those of a $\mathbb{Z}$-form Heis_k for a central

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reduction of the universal enveloping algebra $U(\mathfrak{h})$ of the infinite-dimensional Heisenberg Lie algebra. By [9, 14], there is an injective ring homomorphism

$$\gamma_k : \mathrm{Heis}_k \to K_0(\mathrm{Kar}(\mathcal{H}eis_k)) \tag{1.1}$$

to the Grothendieck ring of the additive Karoubi envelope of $\mathcal{H}eis_k$. In this paper, we prove that γ_k is also surjective, so that $\mathcal{H}eis_k$ *categorifies* Heis_k, as was conjectured in [9, 14]. We also take a first step towards categorification of the comultiplication on $U(\mathfrak{h})$.

To give more precise statements, we need to recall some basic notions. Let Sym be the ring of *symmetric functions*; see [13]. It is freely generated either by the *elementary symmetric functions* $\{e_n\}_{n\geq 1}$ or the *complete symmetric functions* $\{h_n\}_{n\geq 1}$. We also have the *power sums* $\{p_n\}_{n\geq 1}$ whose images generate $\mathrm{Sym}_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Sym}$. Moreover, Sym is a Hopf ring with comultiplication $\delta : \mathrm{Sym} \to \mathrm{Sym} \otimes_{\mathbb{Z}} \mathrm{Sym}$, $f \mapsto \sum_{(f)} f_{(1)} \otimes f_{(2)}$ satisfying

$$\delta(h_n) = \sum_{r=0}^{n} h_{n-r} \otimes h_r, \quad \delta(e_n) = \sum_{r=0}^{n} e_{n-r} \otimes e_r, \quad \delta(p_n) = p_n \otimes 1 + 1 \otimes p_n, \tag{1.2}$$

where $h_0 = e_0 = 1$ by convention. As a $\mathbb{Z}$-module, Sym is free with the canonical basis $\{s_\lambda\}_{\lambda \in \mathcal{P}}$ of *Schur functions* indexed by the set $\mathcal{P}$ of all partitions.

The *infinite-dimensional Heisenberg Lie algebra* is the Lie algebra $\mathfrak{h}$ over $\mathbb{Q}$ with basis $\{c, p_n^{\pm} \mid n \geq 1\}$ and Lie bracket defined from

$$[c, p_n^{\pm}] = [p_m^+, p_n^+] = [p_m^-, p_n^-] = 0, \qquad [p_m^+, p_n^-] = \delta_{m,n} n c. \tag{1.3}$$

The central reduction $U(\mathfrak{h})/(c-k)$ of its universal enveloping algebra may also be realized as the *Heisenberg double* $\mathrm{Sym}_{\mathbb{Q}} \#_{\mathbb{Q}} \mathrm{Sym}_{\mathbb{Q}}$ with respect to the bilinear Hopf pairing

$$\langle -,- \rangle_k : \mathrm{Sym}_{\mathbb{Q}} \times \mathrm{Sym}_{\mathbb{Q}} \to \mathbb{Q}, \quad \langle p_m, p_n \rangle_k = \delta_{m,n} n k. \tag{1.4}$$

By definition, $\mathrm{Sym}_{\mathbb{Q}} \#_{\mathbb{Q}} \mathrm{Sym}_{\mathbb{Q}}$ is the vector space $\mathrm{Sym}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathrm{Sym}_{\mathbb{Q}}$ with associative multiplication defined by

$$(e \otimes f)(g \otimes h) := \sum_{(f),(g)} \langle f_{(1)}, g_{(2)} \rangle_k \, e g_{(1)} \otimes f_{(2)} h.$$

The pairing of two complete symmetric functions is an integer, as follows for example by comparing the coefficients appearing in [1, Th. 5.3] to [1, (2.2)]. Thus we can restrict to obtain a biadditive form $\langle -,- \rangle_k : \mathrm{Sym} \times \mathrm{Sym} \to \mathbb{Z}$. The resulting Heisenberg double

$$\mathrm{Heis}_k := \mathrm{Sym} \#_{\mathbb{Z}} \mathrm{Sym} \tag{1.5}$$

gives us a natural $\mathbb{Z}$-form for $U(\mathfrak{h})/(c-k) \cong \mathrm{Sym}_{\mathbb{Q}} \#_{\mathbb{Q}} \mathrm{Sym}_{\mathbb{Q}}$. For $f \in \mathrm{Sym}$, we write f^- and f^+ for the elements $f \otimes 1$ and $1 \otimes f$ of Heis_k, respectively. Then Heis_k is generated as a ring by the elements $\{h_n^+, e_n^-\}_{n\geq 0}$ subject to the relations

$$h_0^+ = e_0^- = 1, \quad h_m^+ h_n^+ = h_n^+ h_m^+, \quad e_m^- e_n^- = e_n^- e_m^-, \quad h_m^+ e_n^- = \sum_{r=0}^{\min(m,n)} \binom{k}{r} e_{n-r}^- h_{m-r}^+. \tag{1.6}$$

See [1, Section 5] and [12, Appendix A] where this and other presentations are derived. The usual comultiplication on $U(\mathfrak{h})$ descends to ring homomorphisms

$$\delta_{l|m} : \mathrm{Heis}_k \to \mathrm{Heis}_l \otimes_{\mathbb{Z}} \mathrm{Heis}_m, \quad f^{\pm} \mapsto \sum_{(f)} (f_{(1)})^{\pm} \otimes (f_{(2)})^{\pm} \tag{1.7}$$

for $k = l + m$ and $f \in \mathrm{Sym}$.

The antipode induces $\sigma_k : \mathrm{Heis}_k \xrightarrow{\sim} (\mathrm{Heis}_{-k})^{\mathrm{op}}, s_\lambda^\pm \mapsto (-1)^{|\lambda|} s_{\lambda^T}^\pm$. Also there is an isomorphism

$$\omega_k : \mathrm{Heis}_k \xrightarrow{\sim} \mathrm{Heis}_{-k}, \qquad s_\lambda^\pm \mapsto s_{\lambda^T}^\mp. \tag{1.8}$$

The *degenerate Heisenberg category* $\mathcal{H}eis_k$ is a strict $\Bbbk$-linear monoidal category with two generating objects $\uparrow$ and $\downarrow$ and six generating morphisms

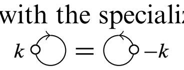
A full set of relations between these generating morphisms is recorded in Definition 5.1 below, where we adopt the usual string calculus for strict monoidal categories. The relations imply that $\mathcal{H}eis_k$ is *strictly pivotal* with duality functor $*$ defined on a morphism by rotating its string diagram through 180°. In particular, the generating objects $\uparrow$ and $\downarrow$ are duals of each other. Letting $\mathfrak{S}_n$ denote the symmetric group with basic transpositions $s_1, \dots, s_{n-1}$, there is also an algebra homomorphism $\imath_n : \Bbbk\mathfrak{S}_n \to \mathrm{End}_{\mathcal{H}eis_k}(\uparrow^{\otimes n})$, which sends s_i to the crossing of the i-th and $(i+1)$-th strings. Note that we always number strings in diagrams by $1, 2, \dots$ from *right to left*.

By the *additive Karoubi envelope* $\mathrm{Kar}(\mathcal{H}eis_k)$ of $\mathcal{H}eis_k$, we mean the idempotent completion of its additive envelope $\mathrm{Add}(\mathcal{H}eis_k)$. Let $K_0(\mathrm{Kar}(\mathcal{H}eis_k))$ be the Grothendieck ring of the monoidal category $\mathrm{Kar}(\mathcal{H}eis_k)$, i.e., the split Grothendieck group with multiplication $[X][Y] := [X \otimes Y]$. For $\lambda \in \mathcal{P}$ with $|\lambda| = n$, let $e_\lambda \in \Bbbk\mathfrak{S}_n$ be the corresponding Young symmetrizer, so that the left ideal $S(\lambda) := (\Bbbk\mathfrak{S}_n)e_\lambda$ is the usual (irreducible) *Specht module* for the symmetric group. Associated to the idempotent e_λ, we also have the object

$$S_\lambda^+ := \left(\uparrow^{\otimes n}, \imath_n(e_\lambda)\right) \in \mathrm{Kar}(\mathcal{H}eis_k). \tag{1.9}$$

Let $S_\lambda^- := (S_\lambda^+)^*$, and set $H_n^\pm := S_{(n)}^\pm$ and $E_n^\pm := S_{(1^n)}^\pm$ for short. Our first main result is as follows.

THEOREM 1.1. – *There is a ring isomorphism* $\gamma_k : \mathrm{Heis}_k \xrightarrow{\sim} K_0(\mathrm{Kar}(\mathcal{H}eis_k))$ *such that* $s_\lambda^\pm \mapsto [S_\lambda^\pm]$ *for each* $\lambda \in \mathcal{P}$. *In particular,* $h_n^\pm \mapsto [H_n^\pm]$ *and* $e_n^\pm \mapsto [E_n^\pm]$. *Also for* $X \in \mathrm{Kar}(\mathcal{H}eis_k)$ *we have that* $[X] = 0 \Rightarrow X = 0$.

This proves extended versions of [9, Conjecture 1] and [14, Conjecture 4.5]. The original conjectures in *loc. cit.* are concerned with the specialization $\mathcal{H}eis_k(\delta)$ of $\mathcal{H}eis_k$ obtained by evaluating the (strictly central) bubble  at a scalar $\delta \in \Bbbk$; see [2, Theorem 1.4]. We will not discuss this specialization further here, but note that our arguments can be carried out in $\mathcal{H}eis_k(\delta)$ in exactly the same way as in $\mathcal{H}eis_k$. Consequently, Theorem 1.1 remains true when $\mathcal{H}eis_k$ is replaced by $\mathcal{H}eis_k(\delta)$. The specialized version with $k = -1, \delta = 0$ or with $k < 0, \delta \in \mathbb{Z}$ proves the original conjectures from [9] and [14], respectively.

The main new ingredient needed to prove Theorem 1.1 is to show that γ_k is *surjective*. We do this by combining the strategy proposed by Khovanov in [9, Section 5] with one additional general result about Grothendieck groups; see Theorem 2.2. This additional result is well known (and easy to prove) in the setting of finite-dimensional algebras. However, we need it here for algebras that are not finite-dimensional and, at this level of generality, we actually could not find it explicitly in the literature (but see [7] for a related result).