ISSN 0012-9593

quatrième série - tome 56 fascicule 5 septembre-octobre 2023

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Amine ASSELAH & Bruno SCHAPIRA

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
Email : annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64. Fax : (33) 04 91 41 17 51
Email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 459 euros.
Abonnement avec supplément papier :
Europe : 646 €. Hors Europe : 730 € ($ 985). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Fabien Durand
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4ᵉ série, t. 56, 2023, p. 1565 à 1582

EXTRACTING SUBSETS MAXIMIZING CAPACITY AND FOLDING OF RANDOM WALKS

BY AMINE ASSELAH AND BRUNO SCHAPIRA

ABSTRACT. – We prove that given any finite set of $\mathbb{Z}^d$, with $d \geq 3$, there is a subset whose capacity and volume are both of the same order as the capacity of the initial set. As an application, we obtain estimates on the probability a transient random walk *covers uniformly* a finite set. Finally, we characterize some *folding* events, under optimal hypotheses. For instance, knowing that a random walk folds to produce an *atypically high occupation density* somewhere, we show that the *folding region* is most likely ball-like, asymptotically as the length of the walk goes to infinity.

RÉSUMÉ. – Nous montrons que de toute partie finie de $\mathbb{Z}^d$, en dimension trois et plus, on peut extraire un sous-ensemble dont la capacité et le volume sont du même ordre de grandeur que la capacité de la partie initiale. Cette observation nous permet d'obtenir, sous des hypothèses optimales, des estimations de la probabilité qu'une marche aléatoire *recouvre uniformément* un ensemble fini. Enfin, nous caractérisons certains évènements de repliement de la marche. Par exemple, lorsque l'on sait qu'une marche aléatoire se replie pour produire une densité d'occupation atypiquement grande, alors la région de repliement a typiquement la forme d'une boule, au sens où sa capacité est du même ordre de grandeur que celle d'une boule.

1. Introduction

This note deals with *capacity* in the context of a random walk on $\mathbb{Z}^d$, with $d \geq 3$. If $\mathbb{P}_x$ is the law of the random walk starting from x, Λ is a non-empty finite subset of $\mathbb{Z}^d$ and H_Λ^+ is the return time into Λ, then the capacity of Λ is

$$\text{cap}(\Lambda) := \sum_{x \in \Lambda} \mathbb{P}_x\big(H_\Lambda^+ = \infty\big). \tag{1.1}$$

Our main observation is that in any finite subset of $\mathbb{Z}^d$, say made of disjoint balls with common radius r, there exists a subset whose size and capacity are both of the same order as the capacity of the initial set. To state precisely this result, let us introduce the needed notation. For $x \in \mathbb{Z}^d$, and $r \geq 1$, we define $B_r(x) = \{z \in \mathbb{Z}^d : \|z - x\| < r\}$, with $\|\cdot\|$ the Euclidean norm, and for $\mathcal{C} \subset \mathbb{Z}^d$, we let $B_r(\mathcal{C}) := \bigcup_{x \in \mathcal{C}} B_r(x)$.

In the whole paper, we deal with space dimension three and higher, and all our results assume this hypothesis.

THEOREM 1.1. – *There exists* $\alpha > 0$*, such that for any* $r \geq 1$ *and any finite* $\mathcal{C} \subset \mathbb{Z}^d$*, there is a subset* $U \subseteq \mathcal{C}$*, satisfying*

$$\text{(i)} \quad \text{cap}(B_r(U)) \geq \alpha \cdot r^{d-2}|U| \quad \textit{and} \quad \text{(ii)} \quad r^{d-2}|U| \geq \alpha \cdot \text{cap}(B_r(\mathcal{C})). \tag{1.2}$$

We now present two applications of this result, Theorems 1.2 and 1.4 below. The former deals informally with the event that a random walk *covers uniformly* a fraction ρ of a set, and bounds the probability of such event by exponential minus ρ times the capacity of the set, under some optimal assumptions on ρ and the scale at which we measure the occupation density. The latter, Theorem 1.4, deals with the shape of the folding region for a walk conditioned on squeezing part of its range, and shows that this region is typically ball-like in the sense that its capacity is of smallest possible order, that is with capacity of order its volume to the power $1 - 2/d$, as it is for balls. This has some natural applications in the context of moderate deviations for the volume or the capacity of the range of the walk, as shown in [1, 2, 3].

Let us also mention that Theorem 1.1 has found application in the context of random interlacements [19, 20].

To be more precise now, for $\Lambda \subset \mathbb{Z}^d$ made of disjoint balls of radius r, consider the event obtained by asking the random walk to spend a time $\rho \cdot r^d$ in each ball making Λ, for some $\rho > 0$. We have shown in [1] how to relate the probability of such covering event with the capacity of Λ. Let $\{S_n\}_{n\in\mathbb{N}}$ denote the discrete-time simple random walk, and $\mathbb{P}$ be its law when starting from the origin. At a time $n \in \mathbb{N} \cup \{\infty\}$, and site $z \in \mathbb{Z}^d$, the local time reads

$$\ell_n(z) := \sum_{k=0}^{n} \mathbf{1}\{S_k = z\} \quad \text{and for } \Lambda \subset \mathbb{Z}^d,\ \ell_n(\Lambda) := \sum_{z\in\Lambda} \ell_n(z). \tag{1.3}$$

THEOREM 1.2. – *There exist positive constants* A *and* κ*, such that for any* $r \geq 1$ *and* $\rho > 0$ *satisfying*

$$\rho r^{d-2} > A, \tag{1.4}$$

one has for any finite $\mathcal{C} \subset \mathbb{Z}^d$

$$\mathbb{P}\big(\ell_\infty(B_r(x)) > \rho r^d, \quad \forall x \in \mathcal{C}\big) \leq \exp\big(-\kappa \cdot \rho \cdot \text{cap}(B_r(\mathcal{C}))\big). \tag{1.5}$$

The condition (1.4) improves the condition in Proposition 1.7 of [1]: $\rho r^{d-2} > A|\mathcal{C}|^{2/d} \cdot \log(n)$. Eliminating the term $|\mathcal{C}|^{2/d}$ is a serious issue which requires Theorem 1.1. Going to an infinite-time setting is straightforward, and is explained in the proof of Theorem 1.2 in Section 4.2.

Note that (1.4) is optimal since typically a walk spends a time of order r^2 in a ball of radius r, conditionally on visiting it. In the case $r = 1$ (when $B_r(x) = \{x\}$ for all x), we obtain a stronger and more general result. First, the result holds true for any $\rho > 0$ and we show that we can take the constant κ equal to one in (1.5). Furthermore, we can deal with non-uniform covering and general transient walks. We refer to Theorem 4.1 in Section 4 for a precise statement.

REMARK 1.3. – We note that Sznitman obtained results with a similar flavor as Theorem 1.2 in the context of the Gaussian free field (GFF) and in the model of random interlacements, respectively in [17, Corollary 4.4] and [18, Theorem 4.2] (see also [12] for related results).

Our second application deals with finite times. For $r \geq 1$, $\rho > 0$, $n \geq 1$, and $\mathcal{C} \subset \mathbb{Z}^d$ finite, we consider

$$\mathcal{F}_n(r,\rho,\mathcal{C}) := \{\forall x \in \mathcal{C}, \quad \ell_n(B_r(x)) > \rho r^d\}. \tag{1.6}$$

In many folding problems, one central issue is to characterize the size and the shape of the folding region $\mathcal{C}$, which might be random. More precisely, one may consider folding events of the form $\bigcup_{\mathcal{C}} \mathcal{F}_n(r,\rho,\mathcal{C})$, where the union is over all $\mathcal{C} \subseteq [-n,n]^d$, with only a lower bound on their volume, say $|\mathcal{C}| \geq L$, when $B_r(\mathcal{C})$ is made of disjoint balls. Then Theorem 1.2 and a naive union bound give

$$\mathbb{P}(\bigcup_{\mathcal{C}} \mathcal{F}_n(r,\rho,\mathcal{C})) \leq (2n+1)^{d\cdot L} \cdot \exp(-\kappa\rho \cdot c \cdot r^{d-2} L^{1-2/d}),$$

using the lower bound on capacity (2.5). The bound just obtained is useful only when

$$\rho \cdot r^{d-2} \geq C L^{2/d} \cdot \log(n). \tag{1.7}$$

Now Theorem 1.1 allows to go beyond this condition (1.7), and gives

$$\mathbb{P}(\bigcup_{\mathcal{C}} \mathcal{F}_n(r,\rho,\mathcal{C})) \leq \exp(-\kappa\rho \cdot c \cdot r^{d-2} L^{1-2/d}),$$

under the weaker assumption:

$$\rho \cdot r^{d-2} \geq C \log(n).$$

The latter is of crucial importance in [3], and can also be used to characterize the shape of a localization region for a random walk, which we now describe in details. First, we introduce more notation. To obtain a neat partition of $\mathbb{Z}^d$ we switch to cubes, rather than balls. Define for $r \geq 1$, and $x \in \mathbb{Z}^d$,

$$Q_r(x) := [x - r/2, x + r/2)^d \cap \mathbb{Z}^d.$$

Define further for $\rho > 0$ and $n \geq 1$,

$$\mathcal{C}_n(r,\rho) := \{x \in r\mathbb{Z}^d \ : \ \ell_n(Q_r(x)) > \rho r^d\}, \quad \text{and} \quad \mathcal{V}_n(r,\rho) := \bigcup_{x \in \mathcal{C}_n(r,\rho)} Q_r(x). \tag{1.8}$$

We can now state our third result.

THEOREM 1.4. – *There are positive constants $\underline{\kappa}$, $\overline{\kappa}$, and C, such that for any $n \geq 2$, r and L positive integers and $\rho > 0$, satisfying*

$$\rho r^{d-2} \geq C \cdot \log(n), \quad \textit{and} \quad n \geq C\rho r^d L, \tag{1.9}$$

one has

$$\exp\big(-\underline{\kappa} \cdot \rho \cdot r^{d-2} \cdot L^{1-2/d}\big) \leq \mathbb{P}\big(|\mathcal{C}_n(r,\rho)| > L\big) \leq \exp\big(-\overline{\kappa} \cdot \rho \cdot r^{d-2} \cdot L^{1-2/d}\big). \tag{1.10}$$

In addition there exists $A > 0$, such that

$$\lim_{n\to\infty} \inf_{(r,\rho,L)} \mathbb{P}\big(\mathrm{cap}(\mathcal{V}_n(r,\rho)) \leq A \cdot |\mathcal{V}_n(r,\rho)|^{1-2/d} \ \big| |\mathcal{C}_n(r,\rho)| > L\big) = 1, \tag{1.11}$$

where the infimum is taken over all triples (r,ρ,L) satisfying (1.9).