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André Belotto DA SILVA & Edward BIERSTONE

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Annales Scientifiques de l'École Normale Supérieure,
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Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
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MONOMIALIZATION OF A QUASIANALYTIC MORPHISM

BY André Belotto DA SILVA AND Edward BIERSTONE

ABSTRACT. – We prove a monomialization theorem for mappings in general classes of infinitely differentiable functions that are called quasianalytic. Examples include Denjoy-Carleman classes, the class of $\mathcal{C}^\infty$ functions definable in a polynomially bounded o-minimal structure, as well as the classes of real- or complex-analytic functions, and algebraic functions over any field of characteristic zero. The monomialization theorem asserts that a mapping in a quasianalytic class can be transformed to a mapping whose components are monomials with respect to suitable local coordinates, by sequences of simple modifications of the source and target—local blowing-ups and power substitutions in the real cases, in general, and local blowing-ups alone in the algebraic or analytic cases. Monomialization is a version of resolution of singularities for a mapping. We show that it is not possible, in general, to monomialize by global blowing-ups, even in the real-analytic case.

RÉSUMÉ. – On démontre un théorème de monomialisation pour les morphismes dans une classe de fonctions quasi-analytiques. Ces classes comprennent, par exemple, les classes de Denjoy-Carleman, la classe des fonctions $\mathcal{C}^\infty$ définissables dans une structure o-minimale polynomialement bornée, ainsi que les classes des fonctions analytiques réelles ou complexes, et de fonctions algébriques sur un corps de caractéristique nulle. Le théorème de monomialisation affirme qu'on peut transformer un morphisme dans une classe quasi-analytique en un autre morphisme dont les composantes sont des monômes dans des coordonnées locales convenables, par une suite de modifications de la source et du but. Dans le cas réel général, celles-ci sont des éclatements locaux et ramifications; dans le cas analytique ou algébrique, elles sont simplement des éclatements locaux. On montre qu'il n'est pas possible, en général, de monomialiser par des éclatements globaux, même dans le cas analytique réel.

1. Introduction

The subjects of this article are monomialization and relative desingularization theorems for mappings in general classes that are called quasianalytic. Quasianalytic classes are classes of infinitely differentiable functions that are characterized by three simple axioms

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including *quasianalyticity*—injectivity of the Taylor series homomorphism at any point—and the implicit function theorem (see Definition 3.1). Examples over $\mathbb{R}$ include the class or real-analytic functions, quasianalytic Denjoy-Carleman classes (objects of study in classical real analysis), and the class of $\mathcal{C}^\infty$ functions which are definable in a given polynomially bounded o-minimal structure (in model theory). The quasianalytic framework covers also complex-analytic mappings, as well as algebraic morphisms (regular morphisms of schemes of finite type over a field $\mathbb{K}$ of characteristic zero) provided that a neighborhood of a point, in this case, is understood to mean an étale neighborhood, so that the implicit function theorem holds.

We prove that a mapping in a real quasianalytic class can be transformed by sequences of simple modifications of the source and target (sequences of local blowing-ups and local power substitutions) to a mapping whose components are monomials with respect to suitable local coordinate systems; see Theorems 1.5 and 1.11. In the general algebraic and analytic cases, we show, moreover, that only local blowing-ups are needed for monomialization (Theorems 1.6 and 1.12).

Our results are best possible in the real cases, because of the following two phenomena. First, we show that it is not possible, in general, to monomialize a proper mapping by global blowing-ups of the source and target, even in the real-analytic case (see Theorem 1.7 and Example 2.4, which is related to [19, Sect. 3]; there is no such counterexample for algebraic or proper complex-analytic morphisms). Secondly, we show that the use of power substitutions is necessary in general quasianalytic classes because of a phenomenon that does not occur in the classes of algebraic or analytic functions—a quasianalytic function defined on a half-line may admit no extension to a quasianalytic function in a neighborhood of the boundary point (Nazarov, Sodin, Volberg [37]; see §1.3). The quasianalytic continuation theorem of [11] intervenes in our proof of monomialization for real quasianalytic classes, in this context. Applications of the monomialization theorem in quasianalytic geometry and model theory are given in §1.3.

Our results in the algebraic and analytic cases include new proofs of Cutkosky's local monomialization theorems [20, 23]. The latter establish local monomialization along a valuation. A basic difference in our approach is that we give a winning strategy for a version of Hironaka's game in the context of monomialization—Alice wins by applying our method to choose the next local blowing-up at *any* point of the fiber, which her opponent Bob is free to pick on his turn. On the other hand, we use étale-local blowing-ups in the algebraic case (see Definitions 1.9), rather than Zariski-local blowing-ups as in [20, 21]. Cutkosky shows that (along a valuation) the stronger result follows from the étale version; we plan to include this in a later article using methods developed here, but in this paper we prefer to emphasize the quasianalytic framework.

Monomialization is a version of resolution of singularities for a mapping (classical resolution of singularities is the case of target dimension one). The monomialization problem in the algebraic case has an extensive literature, going back at least to Zariski, and including or related to the work of Akbulut and King on mappings of complex surfaces [7], the toroidalization, factorization and semistable reduction results of Abramovich, Adiprasito, Denef, Karu, Liu, Matsuki, Temkin and Włodarczyk [1, 2, 3, 6, 44], and the monomialization theorems of Cutkosky and of Denef [24]. Toroidal and semistable morphisms are, in particular, monomial; the main distinction of monomialization in relation to the preceding results on toroidalization and semistable reduction, is the requirement of concrete transformation of the source and target of a given morphism by blowing-ups with smooth centers. In addition to [20, 23], Cutkosky has proved monomialization by global blowing-ups for dominant projective morphisms in dimension three [22].

Our point of view is rather that of analysis, and algebro-geometric techniques used in the preceding results (for example, valuations, Zariski-Riemann manifold) do not appear in our work, although the preceding articles have had an important impact. Indeed, even basic notions of commutative algebra like flatness are not available in local rings of quasianalytic functions, which are not known to be Noetherian in general. On the other hand, resolution of singularities does hold in quasianalytic classes [15, 16], even for ideals that are not necessarily finitely generated [18, Thm. 3.1]. The axioms for a quasianalytic class are meant to capture the minimal properties of a ring of functions that are needed for desingularization of ideals. In the analytic and algebraic cases, the proofs of our main theorems admit significant simplifications, as we will point out in Section 7.

The main new techniques developed in this article are based on logarithmic derivatives (logarithmic with respect to the exceptional divisor that intervenes on blowing up) that are tangent to the fibers of morphism (see §1.2 and Section 5). In this context, we formulate a theorem on resolution of singularities of an ideal relative to a monomial morphism (see Theorems 1.8 and 1.17) that we will prove together with the monomialization theorems above, within a common inductive scheme. The notion of *log derivations tangent to a morphism* coincides with that of *relative log derivations*, of origin in work of Grothendieck and Deligne (see [39, Ch. IV]), and plays a part here, as well as in [5], which is analogous to the one played by standard logarithmic derivations in the proof of resolution of singularities in [17]. We believe the techniques of this article may be useful for global monomialization in the algebraic and proper complex-analytic cases, perhaps combined with methods of [4, 5] (and [12] in low dimensions).

The paper of Dan Abramovich, Michael Temkin and Jarosław Włodarczyk [5] on relative desingularization is an independent work, of which we learned of a preliminary version in July 2019, before posting our article on arXiv that month. We are grateful to the authors for pointing out connections between some of the techniques in our Sections 4, 5 and logarithmic algebraic geometry, and also to Mikhail Sodin for details of the non-extension phenomenon in [37]. We would like to thank also the reviewers and editors for their very helpful detailed comments.

We begin by stating our main results. Details on the notions of blowing up and power substitutions involved will be given in §1.1 following.

Let $\mathcal{Q}$ denote a quasianalytic class, and let M, N denote manifolds (smooth spaces) of class $\mathcal{Q}$; say $m = \dim M$, $n = \dim N$ (see Section 3). Let $\Phi : M \to N$ denote a mapping of class $\mathcal{Q}$; we will say that Φ is a $\mathcal{Q}$-morphism, or simply a morphism if the class is understood.

Definition 1.1. – *Monomial morphism.* The $\mathcal{Q}$-morphism Φ is *monomial at* a point $a \in M$ if there are coordinate systems (of class $\mathcal{Q}$),

$$(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = (u_1, \ldots, u_r, v_1, \ldots, v_s, w_1, \ldots, w_t),$$
$$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = (x_1, \ldots, x_p, y_1, \ldots, y_q, z_{q+1}, \ldots, z_{s'}), \qquad (1.1)$$