ISSN 0012-9593

quatrième série - tome 56 fascicule 6 novembre-décembre 2023

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Giuseppe NEGRO

A sharpened Strichartz inequality for the wave equation

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
Email : annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64. Fax : (33) 04 91 41 17 51
Email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 459 euros.
Abonnement avec supplément papier :
Europe : 646 €. Hors Europe : 730 € ($ 985). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Fabien Durand
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4ᵉ série, t. 56, 2023, p. 1685 à 1708

A SHARPENED STRICHARTZ INEQUALITY FOR THE WAVE EQUATION

BY GIUSEPPE NEGRO

ABSTRACT. – We disprove a conjecture of Foschi, regarding extremizers for the Strichartz inequality with data in the Sobolev space $\dot{H}^{1/2} \times \dot{H}^{-1/2}(\mathbb{R}^d)$, for even $d \geq 2$. On the other hand, we provide evidence to support the conjecture in odd dimensions and refine his sharp inequality in $\mathbb{R}^{1+3}$, adding a term proportional to the distance of the initial data from the set of extremizers. The proofs use the conformal compactification of the Minkowski space-time given by the Penrose transform.

RÉSUMÉ. – Nous infirmons une conjecture de Foschi concernant les points extrémaux de l'inégalité de Strichartz à données dans l'espace de Sobolev $\dot{H}^{1/2} \times \dot{H}^{-1/2}(\mathbb{R}^d)$, où $d \geq 2$ est pair. En revanche, nous donnons des indications en faveur de sa conjecture en dimension impaire, ainsi qu'une version raffinée de son inégalité optimale sur $\mathbb{R}^{1+3}$, en ajoutant un terme proportionnel à la distance des données initiales de l'ensemble des points extrémaux. Les démonstrations utilisent la compactification conforme de l'espace-temps de Minkowski donnée par la transformation de Penrose.

1. Introduction

We consider solutions u to the wave equation $u_{tt} = \Delta u$, on $\mathbb{R}^{1+d}$ with $d \geq 2$, and initial data $\boldsymbol{u}(0) = (u(0), u_t(0))$. We take such initial data in the Sobolev space of pairs $\boldsymbol{f} = (f_0, f_1)$ with norm defined by

$$\|\boldsymbol{f}\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}(\mathbb{R}^d)} = \left(\|(-\Delta)^{1/4} f_0\|^2_{L^2(\mathbb{R}^d)} + \|(-\Delta)^{-1/4} f_1\|_{L^2(\mathbb{R}^d)}\right)^{1/2}. \tag{1}$$

In 1977, Strichartz [22] proved that there is a positive constant C such that

$$\|u\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{1+d})} \leq C\,\|\boldsymbol{u}(0)\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}(\mathbb{R}^d)}. \tag{2}$$

Foschi [10] proved that, for $d = 3$, the optimal constant is attained when

$$\boldsymbol{u}(0) = (u(0), u_t(0)) = \left((1 + |\cdot|^2)^{-\frac{d-1}{2}}, 0\right), \tag{3}$$

Supported by the ERC grant 277778 and the MINECO grants SEV-2011-0087, SEV-2015-0554, MTM2013-41780-P and MTM2017-85934-C3-1-P (Spain), and by the LAGA, Université Paris 13 (France).

meaning that $\mathcal{A}_d = \|u\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{1+d})}/\|\boldsymbol{u}(0)\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}}$, where $\boldsymbol{u}(0)$ is given by (3), is the smallest possible value for the multiplicative constant C in (2) for $d = 3$; precisely, $\mathcal{A}_3 = \left(\frac{3}{16\pi}\right)^{1/4}$. Foschi conjectured that (3) should extremize in any dimension $d \geq 2$. We will provide evidence to support his conjecture in odd dimensions, however we will disprove it in even dimensions; see the forthcoming Theorem 1.2.

Foschi also characterized the initial data that extremize the Strichartz inequality (2) with $d = 3$. The full set $\mathbf{M}$ is obtained by acting a group of symmetries of the inequality on the data (3). Writing

$$\mathrm{d}(\boldsymbol{f}, \mathbf{M}) = \inf_{\boldsymbol{\phi}\in\mathbf{M}} \|\boldsymbol{f} - \boldsymbol{\phi}\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}}, \tag{4}$$

we will mainly be concerned with the following refinement of Foschi's inequality.

THEOREM 1.1. – *There is a positive constant C such that, for all $u\colon \mathbb{R}^{1+3} \to \mathbb{R}$ satisfying $u_{tt} = \Delta u$,*

$$C\,\mathrm{d}(\boldsymbol{u}(0), \mathbf{M})^2 \leq \mathcal{A}_3^2 \|\boldsymbol{u}(0)\|^2_{\dot{H}^{1/2}\times\dot{H}^{-1/2}} - \|u\|^2_{L^4(\mathbb{R}^{1+3})} \leq \mathcal{A}_3^2\,\mathrm{d}(\boldsymbol{u}(0), \mathbf{M})^2.$$

The upper bound is proved in a more general setting in the following section. The lower bound, on the other hand, requires a much more careful treatment and it will follow from a local version, in which we also obtain the optimal constant.

Brezis and Lieb asked if the sharp Sobolev inequality due to Aubin [1] and Talenti [23] could be sharpened in this way; see [6, question (c)]. This was solved by Bianchi and Egnell [5]. Most relevantly, a sharpening of the Strichartz inequality for the Schrödinger equation with $d = 1$ or 2 is implicit in the work of Duyckaerts, Merle and Roudenko [9], who applied their result to the mass-critical nonlinear Schrödinger equation in the small data regime (see also [11]). Theorem 1.1 has a similar application to the cubic nonlinear wave equation in [19].

In the fourth section we consider the deficit functional ψ, defined as

$$\psi(\boldsymbol{u}(0)) := \mathcal{A}_d^p \|\boldsymbol{u}(0)\|^p_{\dot{H}^{1/2}\times\dot{H}^{-1/2}} - \|u\|^p_{L^p(\mathbb{R}^{1+d})},\ p := 2\frac{d+1}{d-1},$$

so that ψ is zero at the supposed extremizers (3). Now for (3) to be extremizing for the Strichartz inequality (2), it must be critical for ψ in the sense that the first derivative of ψ must also vanish there. We will prove the following result disproving Foschi's conjecture in even dimensions.

THEOREM 1.2. – *The data* (3) *are critical for ψ if and only if $d \geq 2$ is odd.*

The case of spatial dimension $d = 2$ is especially surprising. Indeed, in the aforementioned [10, eq. (46)], Foschi proved that $u_\pm(0) = (1 + |\cdot|^2)^{-1/2}$, which is the same function that appears in (3), is an extremizer of the closely related half-wave estimate $\|u_\pm\|_{L^6(\mathbb{R}^{1+2})} \leq (2\pi)^{-1/6}\|u_\pm(0)\|_{\dot{H}^{1/2}}$, in which $\partial_t u_\pm = \pm i\sqrt{-\Delta}u_\pm$; see also [3].

In the fifth section, we prove the lower bound of Theorem 1.1. For this we must show that a spectral gap, associated with the second derivative of ψ, is positive. This is achieved using the Penrose transform, introduced in the third section. Under this transformation, the extremizing pair (3) is mapped to the constant initial data pair $(1/2, 0)$, enabling explicit computations. Compactness arguments will also be required to extend a local version of Theorem 1.1

to the whole space $\dot{H}^{1/2} \times \dot{H}^{-1/2}$. For this we will require a profile decomposition due to Ramos [21], also presented in the third section.

We end the introduction with a mention of the recent paper [12], in which sharp Strichartz estimates for the wave, the half-wave and the Schrödinger equations are studied by means of spacetime transformations such as the Penrose and the Lens transform.

2. Abstract upper bounds

In this section, X will denote a measure space and H will denote a real or complex Hilbert space, with scalar product $\langle\, ,\rangle$ and norm $\|\cdot\|$.

PROPOSITION 2.1. – *For* $1 < p < \infty$, *let* $S : H \to L^p(X)$ *be a linear bounded operator, with operator norm* $\|S\|$. *Then*

$$\|S\|^2\|f\|^2 - \|Sf\|_p^2 \le \|S\|^2 d(f, M_S)^2, \tag{5}$$

where $M_S = \{v \in H : \|Sv\|_p = \|S\|\|v\|\}$ *and* $d(f, M_S) = \inf_{v\in M_S} \|f - v\|$.

We remark that M_S is never empty, as $0 \in M_S$. Also, if $v \in M_S$ then $\lambda v \in M_S$ for all scalars λ; in particular, either $M_S = \{0\}$ or M_S contains nonzero elements of arbitrarily small norm.

Proof of Proposition 2.1 . – If $S = 0$ or $M_S = \{0\}$, then (5) is trivially true. Otherwise, we let $D = d(f, M_S)$, and for $\varepsilon > 0$ we consider a $v \in M_S$ such that $\|f - v\|^2 \le D^2 + \varepsilon$. By the previous remark, we can assume that $v \neq 0$, which implies that $Sv \neq 0$, because $\|Sv\|_p = \|S\|\|v\| \neq 0$.

Now we write $g = f - v$ and we define, for $t \in \mathbb{R}$,

$$h_1(t) = \|S(v + tg)\|_p^2 - \|Sv\|_p^2,\; h_2(t) = \|S(v + tg)\|_p^2 - \|S\|^2\|v + tg\|^2. \tag{6}$$

As a function of t, each of h_1, h_2 is a difference of two convex functions and hence is left and right differentiable at every point. In addition, both are differentiable at $t = 0$, since $Sv \neq 0$; see, for example, [14, Theorem 2.6]

Now, h_2 has a maximum at zero, so $h_2'(0) = 0$. Since h_1 is convex and $h_1(0) = 0$, we have $h_1(1) \ge h_1'(0)$. Therefore

$$\begin{aligned} h_1(1) = \|Sf\|_p^2 - \|Sv\|_p^2 \ge h_1'(0) = (h_1 - h_2)'(0) = 2\|S\|^2\Re(\langle v, g\rangle) \\ = \|S\|^2(\|v + g\|^2 - \|g\|^2 - \|v\|^2). \end{aligned} \tag{7}$$

Recalling that $v + g = f$ and $\|Sv\|_p = \|S\|\|v\|$, this yields

$$\|Sf\|_p^2 \ge \|S\|^2(\|v + g\|^2 - \|g\|^2) \ge \|S\|^2\|f\|^2 - (D^2 + \varepsilon)\|S\|^2, \tag{8}$$

and since this holds for arbitrary $\varepsilon > 0$, the desired conclusion (5) follows. □

The upper bound in Theorem 1.1 is an immediate consequence of the latter proposition, obtained by specializing the operator S to the wave propagator $S_t: \dot{\mathcal{H}}^{1/2}(\mathbb{R}^3) \to L^4(\mathbb{R}^{1+3})$; see the next section.