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DISPERSIVE DECAY OF SMALL DATA SOLUTIONS FOR THE KDV EQUATION

BY MIHAELA IFRIM, HERBERT KOCH AND DANIEL TATARU

ABSTRACT. – We consider the Korteweg-de Vries (KdV) equation, and prove that small localized data yields solutions which have dispersive decay on a quartic time-scale. This result is optimal, in view of the emergence of solitons at quartic time, as predicted by inverse scattering theory.

RÉSUMÉ. – Dans cet article nous considérons l'équation de Korteweg-de Vries (KdV), et montrons que pour des données petites et localisées, les solutions ont une dynamique dispersive sur une échelle de temps quartique. Ce résultat est optimal, comme le prédit la théorie de la diffusion inverse.

1. Introduction

We consider real solutions for the Korteweg-de Vries equation (KdV)

$$\begin{cases} u_t + u_{xxx} - 6uu_x = 0 \\ u(0) = u_0, \end{cases} \tag{1.1}$$

on the real line. Assuming that the initial data is small and localized, we seek to understand the long time dispersive properties of the solution.

This has been a long term goal of research in this direction. In particular, one natural question to ask is whether, for localized initial data, the solutions to the nonlinear equation exhibit the same dispersive decay as the solutions to the corresponding linear equation. In general this is not the case globally in time, due primarily to two types of nonlinear solutions:

(i) Solitons, which move to the right with constant speed.

(ii) Dispersive shocks, where the nonlinearity acts like a transport term and pushes the dispersive part of the solution to the left.

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This paper combines some earlier work and insight gained by the authors when analyzing global or long time dynamical behavior of solutions to certain models of dispersive equations. Our long term goal is to understand the *soliton resolution conjecture* for the nonlinear Korteweg-de Vries equation (KdV). Historically, solitary waves (water waves which do not disperse for a long time and which move at a constant speed without changing their shape) were first observed and reported by John Scott Russell in a shallow canal. He called such a wave "a wave of translation, in a wave tank." This phenomenon was first explained mathematically by Korteweg and de Vries in [20] in 1895. Solitons represent interesting mathematical objects that influence the long time dynamics of the solutions.

The *soliton resolution conjecture* applies to many nonlinear dispersive equations and asserts, roughly speaking, that any reasonable solution to such equations eventually resolves into a superposition of a dispersive component (which behaves like a solution to the linear equation) plus a number of "solitons." This should only be taken as a guiding principle, as many variations can occur; for instance the number of solitons could be finite or infinite, while the dispersive part might not truly have linear scattering, but instead some modified scattering behavior.

This conjecture was studied in many different frameworks (i.e., for different dispersive equations like for example for the nonlinear Schrödinger equation (NLS), see [25] and references within) and it is known in many perturbative cases in the setting: when the solution is close to a special solution, such as the vacuum state or a ground state, as well as in defocusing cases, where no non-trivial bound states or solitons exist. But it is still almost completely open in non-perturbative situations (in which the solution is large and not close to a special solution) which contain at least one bound state.

Turning our attention to solutions to the KdV equation with small initial data, one can distinguish two stages in the nonlinear evolution from the perspective of soliton resolution. Initially, one expects the solutions to satisfy linear-like dispersive bounds. This stage lasts until nonlinear effects (i.e., solitons and dispersive shocks) begin to emerge. The second stage corresponds to solutions which split into at least two of the following components: a linear dispersive part, a dispersive shock, and a soliton.

In this article we aim to describe the first of the two stages above. To better frame the question, we restate the problem as follows:

QUESTION. – *If $\varepsilon \ll 1$ is the initial data size, then what is the time scale up to which the solution will satisfy linear dispersive decay bounds?*

Our main result identifies the quartic time scale $T_\varepsilon = \varepsilon^{-3}$ as the optimal time scale on which linear dispersive decay for all localized data of size $\leq \varepsilon$. The precise statement of the result is provided in Theorem 1.2 below.

We prove this, and also provide some heuristic reasoning, based on inverse scattering, as to why this result is optimal, in other words that the quartic time scale that marks the earliest possible emergence of either solitons or dispersive shocks. To our knowledge this is the first result that rigorously describes the dispersive decay of the solutions on a quartic time-scale.

1.1. The linear KdV flow

If one removes the nonlinearity and considers instead the linear Korteweg-de Vries equation

$$\begin{cases} u_t + u_{xxx} = 0 \\ u(0) = u_0, \end{cases} \tag{1.2}$$

then the solutions will exhibit Airy type decay. To better understand this bound, it is useful to separate the domain of evolution $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ into three regions (see Figure 1 below):

1. The hyperbolic region
$$H := \{x \lesssim -t^{\frac{1}{3}}\},$$
where one sees an oscillatory, Airy type behavior for the solution, with dispersive decay.
2. The self-similar region
$$S := \{|x| \lesssim t^{\frac{1}{3}}\},$$
where the solution essentially looks like a bump function with $t^{-\frac{1}{3}}$ decay.
3. The elliptic region,
$$E := \{x \gtrsim t^{\frac{1}{3}}\},$$
which is eventually left by each oscillatory component of the solution, and consequently where we have better decay.

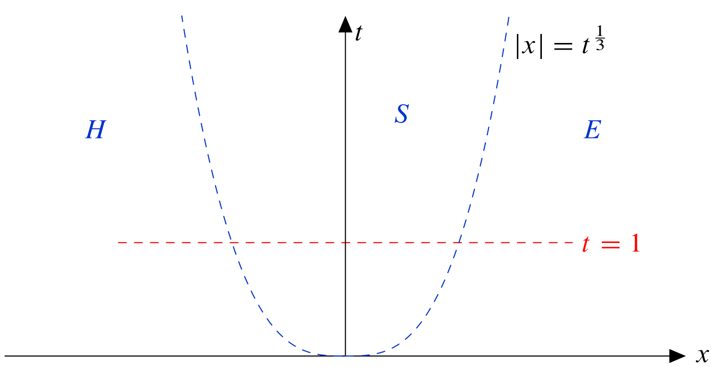

FIGURE 1. The partition into the three main regions: H, E, and S

Consistent with the above partition we define the expression $\langle x \rangle$ in a time dependent fashion as

$$\langle x \rangle := (x^2 + |t|^{\frac{2}{3}})^{\frac{1}{2}}. \tag{1.3}$$

Then the following result describes the dispersive decay of linear KdV waves:

PROPOSITION 1.1. – *Assume that the initial data u_0 for* (1.2) *satisfies*

$$\|u_0\|_{H^1} + \|x^2 u_0\|_{L^2} \leq \varepsilon. \tag{1.4}$$