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A GEOMETRIC APPROACH TO K-HOMOLOGY FOR LIE MANIFOLDS

BY KARSTEN BOHLEN AND JEAN-MARIE LESCURE

ABSTRACT. – We show that the computation of the Fredholm index of a fully elliptic pseudodifferential operator on an integrated Lie manifold can be reduced to the computation of the index of a Dirac operator, perturbed by a smoothing operator, canonically associated, via the so-called clutching map. To this end we adapt to our framework ideas coming from Baum-Douglas geometric K-homology and in particular we introduce a notion of geometric cycles, that can be categorized as a variant of the famous geometric K-homology groups, for the specific situation here. We also define a comparison map between this geometric K-homology theory and a relative K-theory group, directly associated to a fully elliptic pseudodifferential operator.

RÉSUMÉ. – Nous démontrons que le calcul de l'indice de Fredholm d'un opérateur pseudodifférentiel pleinement elliptique sur une variété de Lie intégrée peut être ramené à celui de l'indice d'un opérateur de Dirac perturbé par un opérateur régularisant et canoniquement associé via l'application de serrage (clutching). Pour cela nous adaptons à notre situation des idées venant de la K-homologie géométrique de Baum-Douglas, en particulier nous introduisons une notion de cycles géométriques qui engendrent, dans le contexte spécifique de l'article, un analogue de la K-homologie géométrique bien connue. Nous définissons également une application de comparaison entre notre K-homologie géométrique et un groupe de K-théorie relative directement associé aux opérateurs pseudodifférentiels pleinement elliptiques.

1. Introduction

The Atiyah-Singer index theorem is a celebrated and fundamental result with numerous applications in topology, geometry and analysis. Atiyah and Singer proved the index theorem for elliptic pseudodifferential operators on compact manifolds using K-theory and pseudodifferential operator theory. Later on Atiyah, Bott and Patodi [5] proved the index formula for Dirac type operators using heat kernel methods, and this approach revealed itself very useful for the case of closed smooth manifolds [6, 7, 8, 13, 14, 34]. To recover the full Atiyah-Singer theorem from the special cases covered by heat kernel methods, one needs to reduce the index

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problem for arbitrary elliptic operators to the one for Dirac type operators. Such a reduction is a byproduct of the approach to index theory due to P. Baum and R. Douglas, cf. [10] and [9]. They constructed a geometric K-homology and a suitable comparison homomorphism between the geometric and analytic K-homology groups, and the complete proof that this is in fact an isomorphism was published recently [11]. Thanks to the Baum-Douglas approach to the index theorem, the computation of the Fredholm index of an elliptic pseudodifferential operator on a compact closed manifold can be reduced to the computation of the index of a suitable geometric Dirac operator, naturally associated to a geometric cycle. Motivated by the extension of heat kernel methods to Lie manifolds [14], the purpose of the present work is to address the question of reducing the index problem to the one of Dirac operators for the singular manifolds for which a suitable Lie groupoid allows to well pose the index problem. More precisely, we consider (integrated) Lie manifolds $(M, \mathcal{G})$, that is amenable Lie groupoids $\mathcal{G}$ over compact manifolds with corners such that $M_0 = M \setminus \partial M$ is saturated and $\mathcal{G}_{M_0} = M_0 \times M_0$. For instance, that covers the following situations:

- Manifolds with corners [37]. Here $\mathcal{G} = \mathcal{G}_b$ is obtained after blowing-up successively the submanifolds $H_i \times H_i$ into $M \times M$, where H_i run through the connected boundary hypersurfaces of M, and then removing the so-called lateral faces in b-geometry terminology, which equivalently amounts to consider the subspace $\mathrm{SBlup}_{r,s}(M^2, (H_i^2)_i)$ of the blow-up according to the terminology of [27].
- Manifolds with fibered corners, and thus equivalently stratified pseudomanifolds [25]. Here $\mathcal{G} = \mathcal{G}_\pi$ is obtained as before (blowing-up and removing the lateral faces), but now starting with $\mathcal{G}_b$ in which the fibred diagonals $H_i \times_\pi H_i$ are blown-up in the order prescribed by the order relation between boundary hypersurfaces.
- Manifolds with amenable foliated boundary. The pseudodifferential operators are studied in [45] and the corresponding groupoid $\mathcal{G}_{\mathcal{F}}$, although not directly used, is constructed. Actually, it is stated in [27] that $\mathcal{G}_{\mathcal{F}}$ is obtained by blowing-up in $M \times M$ the holonomy groupoid of the foliation on the boundary, that is $\mathcal{G}_{\mathcal{F}} = \mathrm{SBlup}_{r,s}(M \times M, \mathrm{Hol}(\mathcal{F}))$.
- There are many other examples related to singular spaces, see for instance [40, 41, 19].

In such a case, there is a well defined notion of full ellipticity for operators in the corresponding calculus, that ensures the Fredholmness of the associated operators on M. The question can now be made more precise. Given a fully elliptic operator P on $(M, \mathcal{G})$, can we construct a Dirac operator D in the same calculus, which is Fredholm and with the same index as P? Contrary to the case of C^∞ compact manifolds without boundary, we are not able to give an affirmative answer to this question. Nevertheless, we are able to solve positively the question by allowing tamed Dirac operators, that is, Dirac operators perturbed by smoothing elements in the calculus. Along the way, we prove that if there is no obstruction at the level of K-theory for the full ellipticity (and thus Fredholmness) of Dirac operators, then a perturbation into a Fredholm operator using smoothing operators and elementary Dirac operators always exists. This echoes previous works by Bunke [16] and Carrillo Rouse-Lescure [17]. Explicitly, our main result is the following, cf. Theorem 5.4: Given a pseudodifferential operator P in the pseudodifferential calculus of the integrated Lie manifold $(M, \mathcal{G})$, there is a Callias type operator $\mathcal{C}$ on the clutching Lie manifold $(\Sigma_{\mathcal{A}}, {}^{\varphi}\mathcal{G})$, contained in the relevant calculus, such that the appropriate stable homotopy classes, and thereby the Fredholm indices, of $\mathcal{C}$ and P are equal. The result facilitates a reduction of the index problem for the pseudodifferential operator P to the index computation of the simpler geometric operator $\mathcal{C}$. We note that the literature consists of various index formulas in specialized cases of specific Lie manifolds and for geometric operators of this simpler type. More generally, in conjunction with the recent work of Bohlen-Schrohe [14], the reduction of the index problem to first order geometric operators furnishes a corresponding index formula for fully elliptic pseudodifferential operators on Lie manifolds. Along the way to our main result we prove numerous auxiliary results that are of independent interest. Also, these considerations bring a notion of geometric cycles that mimics the original one of Baum-Douglas with the following main variations into the choice of ingredients:

1. We only accept submersions of manifolds with corners $\varphi : \Sigma \to M$ instead of general continuous maps from Spin^c manifolds to M;
2. We replace Dirac operators with tame Dirac operators.

The point (1) is apparently rather restrictive but actually sufficient for our purpose, since the geometric cycles constructed by the clutching process are of this kind. Furthermore, the classical operations on geometric cycles: isomorphisms, direct sums, cobordisms and vector bundle modifications have a natural analog here. The resulting abelian group, that we call geometric K-homology of $(M, \mathcal{G})$ can then be compared with the suitable relative K-theory group (also known as stable homotopy group of fully elliptic operators). Our approach is definitely tied to the clutching construction which, in a sense, dictated to us the most convenient notion of geometric cycles for our purpose, the latter being, as explained above, the possibility of representing the index of abstract pseudodifferential operators by the one of Dirac operators, perturbed by smoothing operators. We mention that other and more general approaches of geometric K-homology for groupoids exist, in particular in [18, 21, 28]. The observation that one needs to add a regularizing operator to the Dirac operator in order to be able to represent all K-homology classes has appeared before, e.g., in [16, 15, 36, 39], in work on cylindrical ends. In that case, one can absorb the regularizing perturbation into the operator with a suitable change of weight. This approach then leads to obstructions to the existence of analogs of the Atiyah-Patodi-Singer boundary conditions on manifolds with higher dimensional corners, as in [39]. To finish this introduction, let us observe that most bivariant K-theory groups used in this article are of the form $KK_*(\mathbb{C}, A)$. Therefore, the reader unfamiliar with Kasparov bivariant K-theory should keep in mind that only usual K-theory is really needed here. However, it is almost always more convenient for us to have K-theoretic elements represented by Kasparov bimodules rather than by idempotents or unitaries, and hence, unless otherwise stated, $K_*(A)$ will be identified with $KK_*(\mathbb{C}, A)$.

Overview

In Section 2, we study fully elliptic operators contained in the pseudodifferential calculus on the tuple $(M, \mathcal{G})$, where $\mathcal{G}$ is a Lie groupoid over M. We introduce the group of stable homotopy classes of fully elliptic operators ${}^{\mathcal{V}}\mathcal{F}\mathrm{Ell}(M)$ which is defined to equal the relative K-theory group $K(\mu)$, where μ is the homomorphism of the continuous functions M into the full symbol algebra, given by the action as multiplication operators. We prove