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TOWARDS GEOMETRIC SATAKE CORRESPONDENCE FOR KAC-MOODY ALGEBRAS, CHERKIS BOW VARIETIES AND AFFINE LIE ALGEBRAS OF TYPE A

BY HIRAKU NAKAJIMA

ABSTRACT. – We give a provisional construction of the Kac-Moody Lie algebra module structure on the hyperbolic restriction of the intersection cohomology complex of the Coulomb branch of a framed quiver gauge theory, as a refinement of the conjectural geometric Satake correspondence for Kac-Moody algebras proposed in an earlier paper with Braverman, Finkelberg in 2019. This construction assumes several geometric properties of the Coulomb branch under the torus action. These properties are checked in affine type A, via the identification of the Coulomb branch with a Cherkis bow variety established in a joint work with Takayama.

RÉSUMÉ. – Nous donnons une construction provisoire de la structure de module d'algèbre de Lie de Kac-Moody sur la restriction hyperbolique du complexe de cohomologie d'intersection de la branche de Coulomb d'une théorie de jauge avec carquois, comme un raffinement de la correspondance géométrique conjecturale de Satake pour les algèbres de Kac-Moody proposé dans un article antérieur avec Braverman, Finkelberg en 2019. Cette construction suppose plusieurs propriétés géométriques de la branche de Coulomb sous l'action du tore. Ces propriétés sont vérifiées dans le type affine A, via l'identification de la branche Coulomb avec une variété d'arc Cherkis établie dans un travail commun avec Takayama.

Introduction

Let $Q = (Q_0, Q_1)$ be a quiver without edge loops and $\mathfrak{g}_{\mathrm{KM}}$ be the corresponding symmetric Kac-Moody Lie algebra. Let $\mathcal{M}(\lambda,\mu)$ be the Coulomb branch of the framed quiver gauge theory associated with dimension vectors specified by a dominant weight λ and a weight μ with $\mu \leq \lambda$, defined as an affine algebraic variety by Braverman, Finkelberg, and the author [6]. (See also the earlier paper [29] for motivation and references to physics literature.) In the subsequent paper [7, §3(viii)] it was conjectured that there is a geometric construction of an integrable highest weight $\mathfrak{g}_{\mathrm{KM}}$-module structure on the direct sum (over μ) via $\mathcal{M}(\lambda,\mu)$: Recall $\mathcal{M}(\lambda,\mu)$ is equipped with an action of the torus $T = (\mathbb{C}^\times)^{Q_0}$, the Pontryagin dual of the fundamental group of the gauge group, which is $\mathbb{Z}^{Q_0}$ in this case.

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The conjecture in [7] refines and generalizes the earlier conjecture by Braverman-Finkelberg [4], which uses instanton moduli spaces on $\mathbb{R}^4/(\mathbb{Z}/\ell\mathbb{Z})$ in the affine case. The Coulomb branch $\mathcal{M}(\lambda,\mu)$ for an affine type with dominant μ is conjectured to be instanton moduli spaces. (This is proved for affine type A.)

Let Φ denote the hyperbolic restriction functor ([3, 10]) with respect to a generic one parameter subgroup in T. Let us apply it to the intersection cohomology complexes IC of $\mathcal{M}(\lambda,\mu)$ with coefficients in $\mathbb{Q}$. It is conjectured that Φ is hyperbolic semismall in the sense of [5, 3.5.1], and the fixed point set is either empty or a single point. Hence $\mathcal{V}_\mu(\lambda) \overset{\text{def.}}{=} \Phi(\mathrm{IC}(\mathcal{M}(\lambda,\mu)))$ is a vector space. The main part of the conjecture states that $\mathcal{V}(\lambda) = \bigoplus_\mu \mathcal{V}_\mu(\lambda)$ has the structure of an integrable highest weight $\mathfrak{g}_{\mathrm{KM}}$-module $V(\lambda)$ with the highest weight λ so that $\mathcal{V}_\mu(\lambda)$ is a weight space with weight μ. It is regarded as the geometric Satake correspondence for the Kac-Moody Lie algebra $\mathfrak{g}_{\mathrm{KM}}$, as a generalization of the usual geometric Satake for a finite dimensional complex reductive group due to Lusztig, Ginzburg, Beilinson-Drinfeld and Mirković-Vilonen [18, 12, 1, 19]. (See also [11] for a review of the conjecture.) In particular, the hyperbolic restriction functor was used in Miković-Vilonen's realization of representations. Ginzburg's realization of $V(\lambda)$ by the cohomology over the affine Grassmannian has no analog in general $\mathfrak{g}_{\mathrm{KM}}$ setting at the moment when this paper is written.

In this paper, we give a provisional construction of the $\mathfrak{g}_{\mathrm{KM}}$-module structure, assuming several geometric properties of $\mathcal{M}(\lambda,\mu)$. This is a refinement of the conjecture in [7], as well as its supporting evidence since these geometric properties are technical in nature, and not mysterious unlike the $\mathfrak{g}_{\mathrm{KM}}$-module structure. We then check the properties when $\mathfrak{g}_{\mathrm{KM}}$ is of affine type A, using the identification of relevant Coulomb branches with Cherkis bow varieties proved by Takayama and the author [31]. Bow varieties are symplectic reduction, and easier to handle than Coulomb branches. We also use Hanany-Witten transition (see §3(iv)) at various points. It is an isomorphism between two bow varieties. This technique is useful for the following reason. When a bow variety satisfies a balanced condition (see §3(ii)), it is isomorphic to Coulomb branch $\mathcal{M}(\lambda,\mu)$. A fixed point component of a balanced bow variety is another bow variety, but it does not necessarily satisfy the balanced condition. We then show that it is isomorphic to one with the balanced condition by Hanany-Witten transition.

The idea of the construction is simple. The $\mathfrak{g}_{\mathrm{KM}}$-structure should be compatible with restriction to a Levi subalgebra $\mathfrak{l}$, and realized by the hyperbolic restriction functor with respect to a one parameter subgroup $\chi_{\mathfrak{l}}$ corresponding to the Levi subalgebra. When the one parameter subgroup is generic, the Levi subalgebra is Cartan, and we recover the above construction. This compatibility is well-known for the usual geometric Satake correspondence and is a key ingredient of the construction. Therefore we define operators e_i, f_i, h_i corresponding to $i \in Q_0$ by using the hyperbolic restriction with respect to χ_i for the Levi subalgebra $\mathfrak{l}_i$ and the reduction to the case A_1. It is easy to prove the conjecture in the A_1 case. The check of the defining relations on e_i, f_i, h_i, say $[e_i, f_j] = 0$ for $i \neq j$, is reduced to rank 2 cases. By considering tensor products as explained below, it is enough to check them when λ is a fundamental weight. For $\mathfrak{sl}(3)$ relevant bow varieties are affine spaces, and we check them by direct computation. We also realize the embedding $\widehat{\mathfrak{sl}}(n) \to \widehat{\mathfrak{gl}}(\infty)$ by a variant of a family $\underline{\mathcal{M}}(\lambda,\mu)$ below. This argument covers the case $\widehat{\mathfrak{sl}}(2)$. Since we consider only affine types, these are enough.

Unlike in [19] we take $\mathbb{Q}$ as field of coefficients. We believe that some of the arguments survive even in positive characteristic, but we leave the study for the future.

Suppose that Q is of finite type, and hence $\mathfrak{g}_{\mathrm{KM}}$ is a finite dimensional complex simple Lie algebra $\mathfrak{g}$ of type ADE. Then $\mathcal{M}(\lambda,\mu)$ is isomorphic to a transversal slice to an orbit in the closure of another orbit in the affine Grassmannian when μ is dominant [7]. The group $G^\vee$ for the affine Grassmannian is Langlands dual to the simply-connected G with Lie algebra $\mathfrak{g}$. (Hence representations of G are nothing but representations of $\mathfrak{g}$.) This is one of the reasons why we expect the geometric Satake correspondence for $\mathfrak{g}_{\mathrm{KM}}$ via $\mathcal{M}(\lambda,\mu)$. Moreover the hyperbolic restriction $\Phi(\mathrm{IC}(\mathcal{M}(\lambda,\mu)))$ is naturally identified with one in the affine Grassmannian even for non-dominant μ by a recent result by Krylov [16]. Therefore the $\mathfrak{g}$-module structure is induced from the usual geometric Satake correspondence.

From this point of view, our construction above resembles the definition of Kashiwara crystal structure on the set of irreducible components of Mirković-Vilonen cycles by Braverman-Gaitsgory [9]. It is also similar to Vasserot's construction [35] of a $\mathfrak{g}$-module structure. The main difference between these constructions and ours is a construction of an isomorphism between multiplicity spaces appearing in the hyperbolic restriction with respect to χ_i, which we will explain in §1(ii). In our case, the isomorphism is given by the factorization property of Coulomb branches. This isomorphism comes for free or is unnecessary in the usual setting [9, 35]. See §5(viii) for a comparison between our construction and the usual one.

After the author gave a talk on this work at Sydney, B. Webster explained to him an approach to a construction of a $\mathfrak{g}_{\mathrm{KM}}$-module structure via symplectic duality. It is not clear to the author how much can be said in this approach at the time this paper is written. The construction in this paper is nothing to do with the symplectic dual side, which is a quiver variety, where a geometric construction of $\mathfrak{g}_{\mathrm{KM}}$-modules was given in [21, 22]. See [7, §3(viii)] for parallel explanation of two constructions.

After the first version of this paper was written, the author and Weekes generalize the definition of the Coulomb branch of a framed quiver gauge theory to a quiver with *symmetrizer* [32]. Our geometric Satake correspondence is naturally generalized to cover the case when $\mathfrak{g}_{\mathrm{KM}}$ is *symmetrizable* but not necessarily symmetric. The type of the quiver with symmetrizer is Langlands dual $\mathfrak{g}_{\mathrm{KM}}^\vee$ as in the usual geometric Satake correspondence.

The paper is organized as follows. In §1 we formulate conjectures on geometric properties of Coulomb branches under the torus action. In §2 we fix notation for weights of affine Lie algebras. In §3 we review the quiver description and important properties of bow varieties studied in [31]. §4 is the heart of this paper and is devoted to the study of torus action on bow varieties. In §5 we use results in §4 to define a $\mathfrak{g}_{\mathrm{KM}}$ structure on the hyperbolic restriction for affine type A. In the appendix we parametrize torus fixed points in bow varieties when they are smooth. Fixed points are in bijection to Maya diagrams which appear in the infinite wedge space.

Notation

The symmetric group of n letters is denoted by $\mathfrak{S}_n$.