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DYNAMICS OF SCHWARZ REFLECTIONS: THE MATING PHENOMENA

BY Seung-Yeop LEE, Mikhail LYUBICH, Nikolai G. MAKAROV AND Sabyasachi MUKHERJEE

Abstract. – We initiate the exploration of a new class of anti-holomorphic dynamical systems generated by Schwarz reflection maps associated with quadrature domains. More precisely, we study Schwarz reflection with respect to the deltoid, and Schwarz reflections with respect to the cardioid and a family of circumscribing circles. We describe the dynamical planes of the maps in question, and show that in many cases, they arise as unique conformal matings of quadratic anti-holomorphic polynomials and the ideal triangle group.

Résumé. – Nous entamons l'exploration d'une nouvelle classe de systèmes dynamiques anti-holomorphes engendrés par des réflexions de Schwarz associées à des domaines à quadrature. Plus précisément, nous étudions des réflexions de Schwarz par rapport à une deltoïde, par rapport à une cardioide et à une famille de cercle circonscrits. Nous décrivons le plan dynamique des applications en question, et montrons que dans beaucoup de cas, elles sont obtenues à partir d'un unique accouplement conforme d'un polynôme anti-holomorphe quadratique avec le groupe de réflexion d'un triangle idéal.

1. Introduction

Schwarz reflections associated with quadrature domains (or disjoint unions of quadrature domains) provide an interesting class of dynamical systems. In some cases such systems combine the features of the dynamics of rational maps and reflection groups.

A domain in the complex plane is called a quadrature domain if the Schwarz reflection map with respect to its boundary extends meromorphically to its interior. They first appeared in the work of Davis [12], and independently in the work of Aharonov and Shapiro [1, 3, 2]. Since then, quadrature domains have played an important role in various areas of complex analysis and fluid dynamics (see [16] and the references therein).

It is well known that except for a finite number of *singular* points (cusps and double points), the boundary of a quadrature domain consists of finitely many disjoint real analytic curves. Every non-singular boundary point has a neighborhood where the local reflection

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$$\begin{array}{ccc} \overline{\mathbb{D}} & \xrightarrow{\varphi} & \overline{\Omega} \\ {\scriptstyle 1/\overline{z}}\downarrow & & \downarrow{\scriptstyle \sigma} \\ \hat{\mathbb{C}} \setminus \mathbb{D} & \xrightarrow{\varphi} & \hat{\mathbb{C}} \end{array}$$

FIGURE 1. The rational map φ semi-conjugates the reflection map $1/\overline{z}$ of $\mathbb{D}$ to the Schwarz reflection map σ of Ω .

in $\partial\Omega$ is well-defined. The (global) Schwarz reflection σ is an anti-holomorphic continuation of all such local reflections.

Round disks on the Riemann sphere are the simplest examples of quadrature domains. Their Schwarz reflections are just the usual circle reflections. Further examples can be constructed using univalent polynomials or rational functions. Namely, if Ω is a *simply connected* domain and $\varphi : \mathbb{D} \to \Omega$ is a univalent map from the unit disk onto Ω, then Ω is a quadrature domain if and only if φ is a rational function. In this case, the Schwarz reflection σ associated with Ω is semi-conjugate by φ to reflection in the unit circle.

Let us mention two specific examples: the interior of the *cardioid* curve and the exterior of the *deltoid* curve,

$$\left\{ \frac{z}{2} - \frac{z^2}{4} : |z| < 1 \right\} \quad \text{and} \quad \left\{ \frac{1}{z} + \frac{z^2}{2} : |z| < 1 \right\}.$$

In [28], questions on equilibrium states of certain 2-dimensional Coulomb gas models were answered using iteration of Schwarz reflection maps associated with quadrature domains (see [27, §1] for a brief account of this connection). It transpired from their work that these maps give rise to dynamical systems that are interesting in their own right. One of the principal goals of the current paper is to take a closer look at this class of maps and develop a general method of producing conformal matings between groups and anti-polynomials using Schwarz reflection maps associated with disjoint union of quadrature domains. In particular, we will prove that the Schwarz reflection map of the deltoid is a mating of the *ideal triangle group* and the anti-polynomial $\overline{z}^2$.

The *ideal triangle group* $\mathcal{G}$ is generated by the reflections in the sides of a hyperbolic triangle Π in the open unit disk $\mathbb{D}$ with zero angles. Denoting the anti-Möbius reflection maps in the three sides of Π by ρ_1, ρ_2, and ρ_3, we have

$$\mathcal{G} = \langle \rho_1, \rho_2, \rho_3 : \rho_1^2 = \rho_2^2 = \rho_3^2 = \mathrm{id} \rangle < \mathrm{Aut}(\mathbb{D}).$$

Π is a fundamental domain of the group. The tessellation of $\mathbb{D}$ by images of the fundamental domain under the group elements are shown in Figure 2. In order to model the dynamics of Schwarz reflection maps, we define a map

$$\rho : \mathbb{D} \setminus \mathrm{int}\, \Pi \to \mathbb{D}$$

by setting it equal to ρ_k in the connected component of $\mathbb{D} \setminus \mathrm{int}\, \Pi$ containing $\rho_k(\Pi)$ (for $k = 1, 2, 3$). The map ρ extends to an orientation-reversing double covering of $\mathbb{T} = \partial\mathbb{D}$

admitting a Markov partition $\mathbb{T} = [1, e^{2\pi i/3}] \cup [e^{2\pi i/3}, e^{4\pi i/3}] \cup [e^{4\pi i/3}, 1]$ with transition matrix

$$M := \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

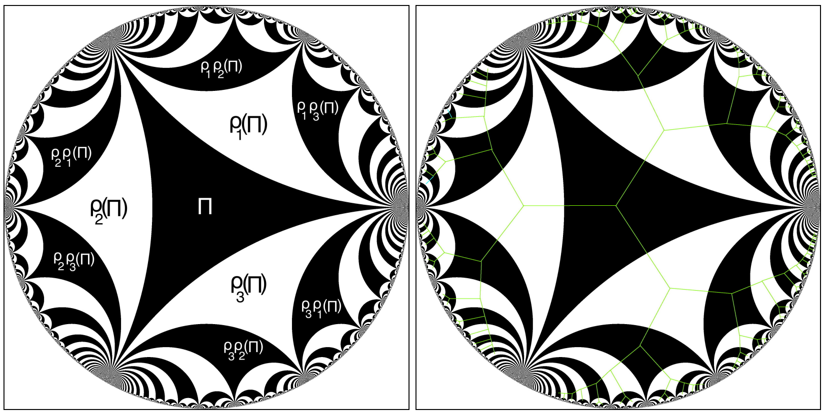

FIGURE 2. Left: The $\mathcal{G}$-tessellation of $\mathbb{D}$ and the formation of a few initial tiles are shown. Right: The dual tree to the $\mathcal{G}$-tessellation of $\mathbb{D}$.

Since the Schwarz reflection maps (which are anti-holomorphic) studied in this paper have a unique, simple critical point, it is not surprising that their dynamics is closely related to the dynamics of quadratic anti-holomorphic polynomials (anti-polynomials for short). The dynamics of quadratic anti-polynomials and their connectedness locus, the Tricorn, was first studied in [11] (note that they called it the Mandelbar set). Their numerical experiments showed structural differences between the Mandelbrot set and the Tricorn. However, it was Milnor who first observed the importance of the Tricorn; he found little Tricorn-like sets as prototypical objects in the parameter space of real cubic polynomials [33], and in the real slices of rational maps with two critical points [34]. Since then, dynamics of anti-holomorphic polynomials and the topological structure of the associated connectedness loci (in particular, the Tricorn) have been studied by various people. We refer the readers to [27, §2] for a survey on this topic.

The connection between quadratic anti-polynomials and the ideal triangle group comes from the fact that the anti-doubling map

$$m_{-2} : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z},\ \theta \mapsto -2\theta$$

(which models the 'external' dynamics of quadratic anti-polynomials) and the map ρ described above admit the same Markov partition with the same transition matrix. This allows one to construct a circle homeomorphism $\mathcal{E} : \mathbb{T} \to \mathbb{T}$ that conjugates the reflection map ρ to the anti-doubling map m_{-2}. The conjugacy $\mathcal{E}$, which is a version of the Minkowski question mark function, serves as a connecting link between the dynamics of Schwarz reflections and that of quadratic anti-polynomials (see the article by Shaun Bullett in [5, §7.8] for a detailed exposition of the Minkowski question mark function, and Subsection 4.4.2 for an