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REGULARITY OF ELLIPTIC AND PARABOLIC SYSTEMS

BY Tobias HOLCK COLDING AND William P. MINICOZZI II

ABSTRACT. – We show uniqueness of cylindrical blowups for mean curvature flow in all dimensions and all codimensions. Cylindrical singularities are known to be the most important; they are the most prevalent in any codimension. Mean curvature flow in higher codimension is a nonlinear parabolic system where many of the methods used for hypersurfaces do not apply. Our results imply regularity of the singular set for the system.

RÉSUMÉ. – Nous montrons un résultat d'unicité pour les profils d'explosion du flot de courbure moyenne, lorsqu'un de ces profils est cylindrique, et cela en toute dimension et codimension. Les singularités cylindriques sont connues pour être les plus importantes et sont les plus répandues en toute codimension. Le flot de courbure moyenne en codimension supérieure est un système parabolique non linéaire, pour lequel la plupart des méthodes utilisées pour les hypersurfaces ne s'appliquent pas. Nos résultats impliquent la régularité du lieu singulier des solutions de ces équations.

0. Introduction

A mean curvature flow is a one-parameter family M_t of n-dimensional submanifolds in $\mathbf{R}^N$ evolving with velocity given by the mean curvature vector. The flow is a nonlinear system of evolution equations. To understand the flow, one needs to understand the singularities that form. The starting point is a blowup analysis where one zooms in near a singularity on smaller and smaller scales, extracting a subsequence that converges to a limit flow. A priori the blowup depends on the choice of the subsequence; whether this occurs is the question of uniqueness of blowups. This is one of the most fundamental issues about singularities, it has been studied in many geometric problems, and has major implications for regularity.

We show that blowups (tangent flows) are unique at each cylindrical singularity of a mean curvature flow (MCF) in arbitrary codimension. A singular point is cylindrical if at least

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one tangent flow is a multiplicity one cylinder $\mathbf{S}^k_{\sqrt{2k}} \times \mathbf{R}^{n-k}$. Uniqueness for cylinders was a major open problem for decades. For hypersurfaces, this problem was solved in [15], but the approach relied heavily on it being a hypersurface and the general case for systems remained an open problem.

THEOREM 0.1. – Let M^n_t be a MCF in $\mathbf{R}^N$. At each cylindrical singular point the tangent flow is unique. That is, any other tangent flow is also a cylinder with the same $\mathbf{R}^k$ factor that points in the same direction.

Cylinders are the most significant singularities because they are the most prevalent in any codimension. By White's parabolic Almgren-Federer dimension reduction, [35], the singular set is stratified into subsets whose Hausdorff dimension is the dimension of translation invariance of the blowup. Thus, for multiplicity one flows, the most prevalent singularities are points where the blowup is $\gamma \times \mathbf{R}^{n-1}$ for a shrinking curve γ. By uniqueness of solutions to ODEs, γ is a closed planar Abresch-Langer curve that is a round circle if embedded or stable, [12], or with low entropy (1) $\lambda(\gamma) < 2$. If the initial data has low entropy $\lambda < 2$, then monotonicity guarantees that every blowup is multiplicity one and, further, the top strata of the singular set consists of cylindrical singularities. If the initial data satisfies the pinching condition of Baker, [5], then every singularity is cylindrical as well.

Using the level set method, MCF of hypersurfaces can be reformulated as a single degenerate parabolic PDE. In the mean convex case, where the flow moves monotonically inward, this simplifies and becomes a degenerate elliptic equation for the arrival time. In these cases, [8] and [23] constructed weak solutions in the viscosity sense and proved that they were Lipschitz. Easy examples show that the solutions do not have to be smooth, but the optimal regularity remained an open question for almost three decades. This was settled in [16]: the arrival time is twice differentiable everywhere and, thus, the viscosity solutions are in fact classical. This is optimal since there are examples where the second derivatives exist but are not even continuous, let alone differentiable, [27]. The crucial ingredient in this sharp analytic result was geometric: uniqueness of cylindrical blowups for MCF hypersurfaces. This may appear surprising, but even for a function on $\mathbf{R}$ differentiability can be thought of as a question of uniqueness: it is differentiable if it has a unique linear approximation independent of scale. The results in this paper should lead to similar regularity for systems and we think, therefore, of the results as "regularity for elliptic and parabolic systems."

There had been two general methods for proving uniqueness, but neither applies here. The earlier general methods go back to Allard-Almgren [1] and Simon [32], and most uniqueness results use one of these. To explain their approach, consider a minimal variety where the blowup is a cone and the intersection of this cone with a unit sphere is a minimal variety Σ in a sphere; Σ is the analog of the time -1 slice of a tangent flow in MCF. Thus, there are spherical slices of the original variety near the singularity that are close to Σ and can be written as entire graphs over Σ, making it possible to linearize the problem. The main obstacle for uniqueness is that Σ has non-trivial Jacobi fields; [1] and [32] are able to deal with this in various settings where Σ is smooth and compact.

(1) The entropy λ is a geometric invariant that is monotone non-increasing under MCF; see (0.4).

In MCF, the blowup is a flow that evolves by dilations and the analog of the link Σ is a time slice called a shrinker. At a cylindrical singularity, the shrinker is a cylinder and is non-compact. When the shrinker is smooth and compact, Schulze [31] proved uniqueness of blowups using the approach of [32]. However, for MCF, as well as many other important equations, the most important singularities are non-compact. The non-compactness means that a flow starting at a closed submanifold is never a graph over the entire non-compact limit. This makes it difficult to linearize the problem, or even to make sense of a neighborhood of the limit, and creates serious analytic issues.

The new approach in [15] to prove uniqueness in codimension one was divided into two main steps – extension and improvement – applied iteratively. The iteration scheme showed that the flow was converging to the limit on ever larger sets as time evolves, giving a way to deal with non-compact singularities. In principle, this scheme can apply to a variety of problems with non-compact limits, as long as one can establish extension and improvement estimates. The extension was quite general, relying largely on monotonicity and Allard/Brakke methods that hold in any codimension. The improvement step was more complicated, quite delicate, and relied on a quantitative form of the classification of hypersurface shrinkers with positive mean curvature. This classification is a type of Bernstein theorem for shrinkers. No such classification holds in higher codimension, just as there is no Bernstein theorem for minimal submanifolds in higher codimension. Entirely new ideas are needed to attack uniqueness for cylindrical blowups in higher codimension. We do this here.

Huisken's seminal paper [25] gave the first uniqueness of blowups for MCF, showing uniqueness of spherical singularities for hypersurfaces. Uniqueness for spheres in higher codimension came much later in the work of Andrews-Baker, [4]. The first general uniqueness result in the non-compact setting was the codimension one cylindrical uniqueness of [15] (cf. [24] when the flow is a graph over the entire cylinder); the arguments relied heavily on properties of hypersurfaces. Since it was already known that mean convex singularities are cylindrical, it followed that blowups are always unique in this setting. In a recent paper, Chodosh and Schulze, [9], proved uniqueness for asymptotically conical shrinkers.

The papers [14, 20] proved strong rigidity for cylinders as shrinkers, [14] for hypersurfaces and then [20] for any codimension; cf. [21]. One consequence of [14, 20] is that all the blowups at a cylindrical singular point must also be cylinders. However, uniqueness and its powerful consequences for the structure of the singular set, require that the possible blowups have the same axis. This uniqueness of the axis in any codimension is the main result in this paper; the arguments here will not use the strong rigidity of the cylinder.

0.1. Key difficulties in higher codimension

It is well-known that higher codimension for MCF creates many new challenges as it does for minimal varieties, [2, 22]. There are many phenomena for hypersurfaces that simply do not extend to higher codimension. New challenges include the vector-valued second fundamental form, a new curvature term P that causes major difficulties, and the lack of a maximum principle. Moreover, the second fundamental form evolves by a reaction-diffusion system in which the reaction terms are very complicated (see Andrews-Baker, [4]), while they are quite easily understood for hypersurfaces.