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PÁL ERDÖS
ISTVÁ N JOÓ
VILMOS KOMORNIK

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CHARACTERIZATION OF THE UNIQUE EXPANSIONS $1 = \sum_{i=1}^{\infty} q^{-n_i}$ AND RELATED PROBLEMS

BY

PÁL ERDÖS, ISTVÁN JOÓ and VILMOS KOMORNIK (*)

RÉSUMÉ. — On caractérise les développements uniques de 1 en bases non entières. On donne une estimation pour la longueur des chiffres 0 consécutifs dans les développements gloutons. On établit certains relations entre ces propriétés et les nombres de Pisot.

ABSTRACT. — We characterize the unique expansions in non-integer bases. We estimate the length of consecutive 0 digits in the greedy expansions. We obtain some relations between these properties and the Pisot numbers.

0. Introduction

Consider a number $1 < q < 2$. By an expansion of a real number x we mean a representation of the form

$$x = \sum_{i=1}^{\infty} \varepsilon_i q^{-i}, \quad \varepsilon_i \in \{0, 1\}.$$

It is clear that x has an expansion if and only if $0 \le x \le 1/(q-1)$.

Let us introduce the lexicographic order $\overset{L}{<}$ between the real sequences : $(\varepsilon_i) \overset{L}{<} (\varepsilon'_i)$ if there is a positive integer m such that $\varepsilon_i = \varepsilon'_i$ for all $i < m$ and $\varepsilon_m < \varepsilon'_m$. It is easy to verify that for every fixed $0 \le x \le 1/(q-1)$

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P. ERDÖS, Mathematical Institute of the Hungarian Academy of Sciences, Reáltanoda u. 13–15, 1053 Budapest, Hongrie.
I. JOÓ, Eötvös Loránd University, Dept. of Analysis, Múzeum krt. 6-8, 1088 Budapest, and Mathematical Institute of the Hungarian Academy of Sciences, Reáltanoda u. 13–15, 1053 Budapest, Hongrie.
V. KOMORNIK, Eötvös Loránd University, Dept of Analysis, Múzeum krt. 6-8, 1088 Budapest, Hongrie et Univ. de Bordeaux I, UFR de Mathématiques, 351 cours de la Libération, 33405 Talence Cedex, France.

in the set of all expansions of x there is a greatest and a smallest element with respect to this order : the so-called *greedy* and *lazy* expansion, *cf.* [4]. (The greedy expansions were studied earlier in [1] where they were called β-expansions.) A number x has a unique expansion if and only if its greedy and lazy expansions coincide.

Let us recall that the digits of these expansions may be defined recursively as follows : if $m \geq 1$ and if the digits ε_i of the greedy expansion of x are defined for all $i < m$, then we put

$$\varepsilon_m = \begin{cases} 1 & \text{if } \sum_{i<m} \varepsilon_i q^{-i} + q^{-m} \leq x, \\ 0 & \text{if } \sum_{i<m} \varepsilon_i q^{-i} + q^{-m} > x. \end{cases}$$

If $m \geq 1$ and if the digits ε_i of the lazy expansion of x are defined for all $i < m$, then we put

$$\varepsilon_m = \begin{cases} 0 & \text{if } \sum_{i<m} \varepsilon_i q^{-i} + \sum_{i>m} q^{-i} \geq x, \\ 1 & \text{if } \sum_{i<m} \varepsilon_i q^{-i} + \sum_{i>m} q^{-m} < x. \end{cases}$$

In section 1 we characterize the unique expansions of 1. This improves some earlier results in [5]. As a by-product we obtain a new proof for the characterization of the greedy expansions, obtained earlier in [2].

In [4] it was proved that for almost every $1 < q < 2$ the greedy expansion of 1 contains arbitrarily long sequences of consecutive 0 digits. In section 2 we improve this result by giving an explicit estimate on the length of these sequences. An analogous result is obtained for the lazy expansions, too.

In section 3 we generalize some other results obtained in [4]–[7].

At the end of this paper we formulate some open questions.

The authors wish to thank the referee for drawing their attention to the papers [2], [3] and [9].

1. Characterization of the greedy and the unique expansions of 1

Fix $1 < q < 2$ arbitrarily and consider an expansion of 1 :

$$1 = \sum_{i=1}^{\infty} \varepsilon_i q^{-i}, \qquad \varepsilon_i \in \{0, 1\}. \tag{1}$$

THEOREM 1

a) (1) *is the greedy expansion of* 1 *if and only if*

$$(\varepsilon_{k+i}) \overset{\mathrm{L}}{<} (\varepsilon_i) \quad \text{whenever } \varepsilon_k = 0. \tag{2}$$

b) (1) *is the unique expansion of* 1 *if and only if* (2) *and*

$$(1-\varepsilon_{k+i}) \overset{\mathrm{L}}{<} (\varepsilon_i) \quad \textit{whenever } \varepsilon_k = 1. \tag{3}$$

are satisfied. □

Remark 1. — It is easy to deduce from this theorem that if (1) is the greedy (resp. unique) expansion of 1, then (2) (resp. (2) and (3)) is satisfied for all $k \geq 1$. □

The proof of this theorem is based on some lemmas concerning the more general expansions

$$x = \sum_{i=1}^{\infty} \varepsilon_i q^{-i}, \quad \varepsilon_i \in \{0,1\} \tag{4}$$

for arbitrarily fixed $1 < q < 2$ and $0 \leq x \leq 1/(q-1)$.

LEMMA 1

a) (4) *is the greedy expansion of x if and only if*

$$\sum_{i=1}^{\infty} \varepsilon_{k+i} q^{-i} < 1 \quad \textit{whenever } \varepsilon_k = 0. \tag{5}$$

b) (4) *is the lazy expansion of x if and only if*

$$\sum_{i=1}^{\infty} (1-\varepsilon_{k+i}) q^{-i} < 1 \quad \textit{whenever } \varepsilon_k = 1. \tag{6}$$

Proof :

a) If (5) is not satisfied for some $\varepsilon_k = 0$, then x has another expansion

$$x = \sum_{i=1}^{\infty} \varepsilon'_i q^{-i}, \quad \varepsilon'_i \in \{0,1\} \tag{7}$$

such that $\varepsilon_i = \varepsilon'_i$ for all $i < k$ and $\varepsilon'_k = 1$. Then the expansion (4) is not greedy.

If the expansion (4) is not greedy, then there is another expansion (7) of x and there is a positive integer k such that $\varepsilon_i = \varepsilon'_i$ for all $i < k$ and $\varepsilon_k = 0$, $\varepsilon'_k = 1$. It follows that

$$\sum_{i>k} \varepsilon_i q^{-i} \geq q^{-k}$$

and therefore (5) is not satisfied.

b) The assertion follows at once from a) if we remark that the expansion (4) is lazy if and only if the expansion

$$1/(q-1) - x = \sum_{i=1}^{\infty} (1-\varepsilon_i) q^{-i} \tag{8}$$

is greedy. ▯

LEMME 2

a) *If* $x \geq 1$ *and if the expansion* (4) *is greedy, then* (2) *is satisfied.*

b) *If* $x \geq 1$ *and if the expansion* (4) *is unique, then* (2) *and* (3) *are satisfied.*

Proof:

a) Assume that (2) is not satisfied for some $\varepsilon_k = 0$, then either $(\varepsilon_{k+i}) = (\varepsilon_i)$ or $(\varepsilon_{k+i}) \overset{\mathrm{L}}{>} (\varepsilon_i)$. In the first case we have

$$\sum_{i=1}^{\infty} \varepsilon_{k+i} q^{-i} = \sum_{i=1}^{\infty} \varepsilon_i q^{-i} = x \geq 1\,;$$

hence the condition (5) of LEMMA 1 is not satisfied and the expansion (4) is not greedy. In the second case there is an integer m such that $\varepsilon_{k+i} = \varepsilon_i$ for all $i < m$ and $\varepsilon_{k+m} = 1$, $\varepsilon_m = 0$. If the expansion (4) were greedy, then by LEMMA 1 we would have

$$\sum_{i=1}^{\infty} \varepsilon_{k+i} q^{-i} < 1 \leq x.$$

Therefore x would have another expansion (7) such that $\varepsilon'_i = \varepsilon_i$ for all $i < m$ and $\varepsilon'_m > \varepsilon_m$; hence $(\varepsilon'_i) \overset{\mathrm{L}}{>} (\varepsilon_i)$. But this is impossible because (4) is the greedy expansion.