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ON PARTIALLY HYPERBOLIC DIFFEOMORPHISMS IN DIMENSION THREE VIA A NOTION OF AUTONOMOUS DYNAMICS

by Souheib Allout & Kambiz Moghaddamfar

Abstract. — We introduce a notion of autonomous dynamical systems and apply it to prove rigidity of partially hyperbolic diffeomorphisms on closed compact three-manifolds under some smoothness hypothesis of their associated framing.

Résumé (*Sur les difféomorphismes partiellement hyperboliques en dimension trois via une notion de systémes dynamiques autonomes*). — Nous introduisons une notion de systèmes dynamiques autonomes et l'appliquons pour prouver une rigidité des difféomorphismes partiellement hyperboliques en dimension trois sous une hypothèse de régularité de leur champs de repères associés.

1. Introduction and statement of main results

The study of partially hyperbolic diffeomorphisms in dimension three, both in the flexible and rigid settings, has been widely explored and it is still an active area of research. Rigidity of these diffeomorphisms, satisfying some additional requirements, has been investigated in both smooth and topological frameworks. In this paper, which is motivated by [8] (and also [18]), we provide

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smooth "rigid" classifications of partially hyperbolic diffeomorphisms satisfying some additional hypothesis. We show then how to extend this to the C^1-regularity case.

1.1. An introductory example. — (see more details in Section 6) The better is perhaps to start with the following construction illustrating the content of the present paper. Let $A : \mathbb{T}^2 \to \mathbb{T}^2$ be a hyperbolic toral linear automorphism. Let $c : \mathbb{T}^2 \to \mathbb{S}^1$ be a smooth map and consider the *skew product* φ on $M = \mathbb{T}^2 \times \mathbb{S}^1$ defined by:

$$\varphi(x, \theta) = (A(x), \theta + c(x))$$

One can construct continuous vector fields X_s, X_u, and X_c on M satisfying $\varphi_*(X_s) = \lambda_s X_s, \varphi_*(X_u) = \lambda_u X_u$ and $\varphi_*(X_c) = \lambda_c X_c$. The vector field X_c corresponds to the $\mathbb{S}^1$ factor, while X_s and X_u correspond to linear vector fields on $\mathbb{T}^2$ and $0 < \lambda_s < 1 < \lambda_u$ are the eigenvalues of A. The existence of X_s and X_u follows from basics of partially hyperbolic dynamics theory which will be the central subject in this paper.

Our map φ is a special partially hyperbolic diffeomorphism in the sense that it has continuous, even constant, Lyapunov exponents (and splittings)! This situation corresponds to our second topic here: autonomous dynamical systems, to mean there exists a framing where the derivative cocycle map is constant. We prove a differentiable rigidity of partially hyperbolic autonomous systems in dimension three as well as autonomous systems in dimension two.

To get a flavour of the results and methods in this paper, let us assume that X_s and X_u are C^1. Their bracket $Z = [X_s, X_u]$ is a well defined C^0 vector field. One observes that $Z = aX_c$ for a φ-invariant function a. But, a being $\mathbb{S}^1$-invariant implies that it is in fact constant. We also have $[X_s, X_c] = [X_u, X_c] = 0$, and therefore, X_s, X_c, X_u generate a three dimensional Lie subalgebra isomorphic to the Heisenberg algebra if $a \neq 0$ and to $\mathbb{R}^3$ otherwise. Thus, M is a quotient of the simply connected Lie group, associated to this subalgebra, by a lattice. By algebraic topological arguments, one excludes the Heisenberg case and, thus, $\mathbb{R}^3$ acts transitively and locally freely on M. One, then, shows that φ is C^1-conjugate to an affine partially hyperbolic automorphism on $\mathbb{T}^3$. Its linear part (as an element of $\mathsf{GL}(3, \mathbb{Z})$) is, up to a finite cover, C^1-conjugate to a direct product:

$$(x, \theta) \in \mathbb{T}^2 \times \mathbb{S}^1 \to (A(x), \theta) \in \mathbb{T}^2 \times \mathbb{S}^1$$

Obviously, this is a special situation that happens for rare maps $c : \mathbb{T}^2 \to \mathbb{S}^1$; even if the C^0-vector fields X_s, X_u and X_c exist for all.

Let us emphasize that our regularity assumption (for vector fields) is merely C^1, and not C^2 as it is usually the case for structures involving a kind of curvature ...! For this it is worthwhile to recall that a C^1 vector field generates a C^1 flow (not just a C^0 one). Also, applying any definition of brackets (for instance as commutators of operators on C^∞ functions), the bracket of two C^1 vector fields is a well defined C^0 vector field.

Actually we think that the previous outlined rigidity extends to the case where the vector fields are Lipschitz?! Indeed, Frobenius' theorem is valid with this regularity ([23]), and therefore also is the theorem of Palais on integration of Lie algebra actions since its proof is based on Frobenius (see appendix 7 for more details).

1.2. After this warming up example, let us give more precise definitions. Let M be a compact smooth manifold of dimension n endowed with a parallelization (framing) $\mathcal{F}$ of its tangent bundle, *i.e.* a system of vector fields defining a basis of each tangent space and let φ be a C^1 diffeomorphism of M. The derivative cocycle is the map:

$$x \in M \to C_\varphi(x) \in \mathsf{GL}(n, \mathbb{R})$$

where $C_\varphi(x)$ is the matrix of the derivative $D_x\varphi : T_xM \to T_{\varphi(x)}M$, when these linear spaces are endowed with the bases $\mathcal{F}(x)$ and $\mathcal{F}(\varphi(x))$ respectively.

We say that φ is *autonomous* with respect to $\mathcal{F}$ if the cocycle map C_φ is constant. The same definition applies to a C^1 action of a group G to mean that any element of it is autonomous with respect to $\mathcal{F}$.

Now, a G-action on M is said to be autonomous if it is autonomous with respect to some parallelization $\mathcal{F}$ on M. In this case we have a representation $G \to \mathsf{GL}(n, \mathbb{R})$ associating to each $g \in G$ the cocycle matrix of g (acting on M). This is essentially inspired by [25] and [24] where the terminology *autonomous* was introduced. Our contribution here is to prove results towards the classification of such autonomous systems in some cases. As one may expect, the regularity of the framing is relevant and it is natural to ask when is it possible to have a constant matrix? i.e. when is it possible for a diffeomorphism to be autonomous?

1.2.1. *Regularity of the framing.* — We will always assume the manifold M and the diffeomorphism φ to be smooth. Our rigidity results require the framing to be at least of class C^1. Continuous framings are also interesting, but we will not deal with them in the present article. The introductory example in Section 1.1 shows how abundant they are. Less than C^0, say measurable framings, seems to be not restrictive.

1.3. Examples. — Here we give some examples of autonomous actions that will be explained in more details in the next section:

1.3.1. *Toral affine automorphisms.* — Consider the space $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. The group $\mathsf{GL}(n, \mathbb{Z}) \ltimes \mathbb{T}^n$ acts naturally on $\mathbb{T}^n$ by affine automorphisms (where $\mathsf{GL}(n, \mathbb{Z})$ denotes the group of integer matrices with determinant ± 1). This