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12. INVARIANTS OF TERNARY QUADRATIC FORMS

by

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Abstract. — This paper deals with Gross–Keating invariants of ternary quadratic forms over $\mathbb{Z}_\ell$. The main technical difficulties arise in residue characteristic $\ell = 2$. In this case, we define the Gross–Keating invariants in terms of a normal form. We give an alternative, less computational approach for anisotropic quadratic forms.

***Résumé* (Invariants de Gross–Keating pour les formes quadratiques ternaires)**
Cet article concerne les invariants de Gross–Keating pour les formes quadratiques ternaires sur $\mathbb{Z}_\ell$. Les difficultés principales n'apparaissent qu'en caractéristique résiduelle $\ell = 2$. Dans ce cas, nous déterminons les invariants de Gross–Keating en termes d'une forme normale. Pour les formes anisotropes nous donnons une approche plus directe.

This note provides details on [**GK**, Section 4]. The main goal is to define and compute the Gross–Keating invariants a_1, a_2, a_3 of ternary quadratic forms over $\mathbb{Z}_\ell$ (Definition 1.2). If $a_1 \equiv a_2 \bmod 2$ and $a_3 > a_2$ we define an additional invariant $\epsilon \in \{\pm 1\}$ (Definition 2.7, Definition 4.8). If $\ell \neq 2$ every quadratic form over $\mathbb{Z}_\ell$ is diagonalizable, and it is easy to determine these invariants from the diagonal form (Section 2). If $\ell = 2$ not every quadratic form is diagonalizable. Moreover, even for diagonal quadratic forms it is not straightforward to determine the Gross–Keating invariants. We determine a normal form in Section 3 and compute the invariants in terms of this normal form (Section 4). In Section 5 we determine explicitly when a ternary quadratic form is anisotropic. A complete table can be found in Proposition 5.2 (non diagonalizable case) and Theorem 5.7 (diagonalizable case). In Section 6, we give an alternative definition of the Gross–Keating invariants for anisotropic quadratic forms. The results of Section 6 are due to Stefan Wewers, following a hint in [**GK**, Section 4].

Our main reference on quadratic forms over $\mathbb{Z}_\ell$ is [**C**, Chapter 8]. Most of the results of this paper can also be found in the work of Yang, in a somewhat different

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form. The Gross–Keating invariants are computed in [**Y1**, Appendix B]. The question whether a given form over $\mathbb{Z}_2$ is isotropic or not (Section 5) is discussed in [**Y2**].

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1. Definition of the invariants a_i

In this section we give the general definition of the Gross–Keating invariants a_i of quadratic forms over $\mathbb{Z}_\ell$ which are used in [**GK**].

Let L be a free $\mathbb{Z}_\ell$-module of rank n and choose a (for the moment) arbitrary basis $\boldsymbol{\psi} = \{\psi_1, \psi_2, \ldots, \psi_n\}$. For the application to [**GK**] we are only interested in the case $n = 3$ of ternary quadratic forms. Let (L, Q) be an integral quadratic form over $\mathbb{Z}_\ell$, that is,

$$Q(x) = Q\Big(\sum x_i\psi_i\Big) = \sum_{i\leq j} b_{ij}x_ix_j, \qquad \text{with } b_{ij} \in \mathbb{Z}_\ell.$$

Put $b_{ji} = b_{ij}$ for $j > i$. If we want to stress the dependence of the b_{ij} on the basis, we write $b_{ij}(\boldsymbol{\psi})$ for b_{ij}. We write $(x, y) = Q(x + y) - Q(x) - Q(y)$ for the corresponding symmetric bilinear form and $B = ((\psi_i, \psi_j))$ for the corresponding matrix. Note that

$$B = \big(B_{ij}\big), \qquad \text{where} \qquad B_{ij} = \begin{cases} b_{ij}, & \text{if } i < j, \\ 2b_{ij}, & \text{if } i = j. \end{cases}$$

In the rest of the paper we only use the b_{ij} and not the B_{ij}, for simplicity. We denote by ord the ℓ-adic valuation on $\mathbb{Z}_\ell$. We always suppose that Q is regular, that is, $\det(B) \neq 0$.

Changing the basis multiplies the determinant of B by an element of $(\mathbb{Z}_\ell^\times)^2$. Therefore the determinant is a well defined element of $\mathbb{Z}_\ell/(\mathbb{Z}_\ell^\times)^2$.

Lemma 1.1. — *Suppose that either $\ell \neq 2$ or n is odd. Define*

$$\Delta = \Delta(Q) = \frac{1}{2}\det(B).$$

Then $\Delta \in \mathbb{Z}_\ell$.

Proof. — The lemma is obvious if $\ell \neq 2$. Suppose that $\ell = 2$ and n odd. Write $\Delta = \sum_{\sigma\in S_n} 2^{\delta(\sigma)}d(\sigma)$, where $d(\sigma) = (-1)^{\mathrm{sgn}(\sigma)}\prod_{i=1}^n b_{i\sigma(i)}$ and $\delta(\sigma)+1$ is the number of $i \in \{1, 2, \ldots, n\}$ which are fixed by σ. The only problematic terms are those with $\delta(\sigma) = -1$. Suppose that σ acts without fixed points on $\{1, 2, \ldots, n\}$. Then $\sigma^{-1} \neq \sigma$, since n is odd. The matrix $((\psi_i, \psi_j))$ is symmetric. It follows that $d(\sigma) = d(\sigma^{-1})$, hence $2^{\delta(\sigma)}d(\sigma) + 2^{\delta(\sigma^{-1})}d(\sigma^{-1}) \in \mathbb{Z}_\ell$. $\square$

We now come to the definition of the Gross–Keating invariants of a quadratic form. Let $\boldsymbol{\psi} = (\psi_1, \psi_2, \ldots, \psi_n)$ be a basis of L. We write $S(\boldsymbol{\psi})$ for the set of tuples $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{Z}^n$ such that

$$(1.1) \qquad y_1 \leq y_2 \leq \cdots \leq y_n, \qquad \frac{y_i + y_j}{2} \leq \mathrm{ord}(b_{ij}(\boldsymbol{\psi})) \quad \text{for } 1 \leq i \leq j \leq n.$$

Let $S = \cup S(\boldsymbol{\psi})$. We order tuples $(y_1, \dots, y_n) \in S$ lexicographically, as follows. For given $(y_1, \dots, y_n)$, $(z_1, \dots, z_n) \in S$, let j be the largest integer such that $y_i = z_i$ for all $i < j$. Then $(y_1, \dots, y_n) > (z_1, \dots, z_n)$ if $y_j > z_j$.

Definition 1.2. — The *Gross–Keating invariants* $a_1, \dots, a_n$ are the maximum of $(y_1, \dots, y_n) \in S$. A basis $\boldsymbol{\psi}$ is called *optimal* if $(a_1, \dots, a_n) \in S(\boldsymbol{\psi})$.

If $\boldsymbol{\psi}$ is optimal, then

$$a_i + a_j \leq 2\,\mathrm{ord}(b_{ij}(\boldsymbol{\psi})) \quad \text{for} \quad 1 \leq i \leq j \leq n, \quad \text{and} \quad a_1 \leq a_2 \leq \dots \leq a_n. \tag{1.2}$$

Since Δ is well defined up to $(\mathbb{Z}_\ell^\times)^2$, the integer $\mathrm{ord}(\Delta)$ is well defined. The following lemma will be useful in computing the Gross–Keating invariants.

Lemma 1.3

(a) *Suppose that n is odd, then*

$$\mathrm{ord}(\Delta) \geq a_1 + a_2 + \dots + a_n.$$

(b) *We have*

$$a_1 = \min_{x,y \in L} \mathrm{ord}\,(x, y)\,.$$

(c) *Define $\rho := \min_A \mathrm{ord}(\det(A))$, where A runs through the 2 by 2 minors of B. Then*

$$a_1 + a_2 \leq \rho.$$

Proof. — This lemma is proved in [**Y1**, Lemma B.1, Lemma B.2]. Note that the matrix T in [**Y1**] differs by a factor 2 from our matrix B. Let $\boldsymbol{\varphi}$ be an optimal basis. We use the notation of the proof of Lemma 1.1.

First suppose that $\ell = 2$. Write $\mathbb{S}$ for the set of equivalence classes in S_n under the equivalence relation $\sigma \sim \sigma^{-1}$. The proof of Lemma 1.1 shows that $\Delta = \sum_{\sigma \in \mathbb{S}} (-1)^{\mathrm{sgn}(\sigma)} 2^{\delta'(\sigma)} d(\sigma)$, where $\delta'(\sigma) \geq 0$. The choice of $\boldsymbol{\varphi}$ implies that

$$\mathrm{ord}(2^{\delta'(\sigma)} d(\sigma)) = \delta'(\sigma) + \mathrm{ord}\Big(\prod_i b_{i\sigma(i)}\Big) \geq \sum_{i=1}^n \frac{a_i + a_{\sigma(i)}}{2} = \sum_{i=1}^n a_i.$$

This proves (a) in this case.

If $\ell \neq 2$, define $\delta'(\sigma) = 0$ for all $\sigma \in S_n$. Then the proof works also in this case.

Since $a_1 \leq a_2 \leq \dots \leq a_n$, it follows from (1.2) that $\mathrm{ord}(b_{ij}(\boldsymbol{\varphi})) \geq a_1$ for all $i \leq j$. On the other hand, it is obvious that $a_1 \geq \min_{x,y \in L} \mathrm{ord}\,(x, y)$. This implies (b).

Part (c) is similar to (a), compare to Lemma B1.ii in [**Y1**]. Let $i_1, i_2, j_1, j_2 \in \{1, 2, \dots, n\}$ be integers such that $i_1 \neq i_2$ and $j_1 \neq j_2$. Write $B(i_1, i_2; j_1, j_2)$ for the corresponding minor of B. After renumbering, we may suppose that $i_1 \neq j_2$ and $i_2 \neq j_1$. Then $\det(B(i_1, i_2; j_1, j_2)) = \pm(2^\alpha b_{i_1,j_1} b_{i_2,j_2} - b_{i_1,j_2} b_{i_2 j_1})$, where $\alpha \in \{0, 1, 2\}$ is the number of equalities $i_1 = j_1, i_2 = j_2$ that hold. We conclude that $\mathrm{ord}(\det(B(i_1, i_2; j_1, j_2)) \geq (a_{i_1} + a_{i_2} + a_{j_1} + a_{j_2})/2 \geq a_1 + a_2$. (Here we use that $a_1 \leq a_2 \leq \dots \leq a_n$ and $i_1 \neq i_2$ and $j_1 \neq j_2$.) This proves (c). □

2. Definition of the Gross–Keating invariants for $\ell \neq 2$

We start this section with an elementary lemma which holds without assumption on ℓ.

Lemma 2.1. — *Choose a basis $\boldsymbol{\psi} = (\psi_1, \dots, \psi_n)$ of L. Let $\gamma_1, \dots, \gamma_m \in L$ be linearly independent. The following are equivalent.*

(a) *There exists $\gamma_{m+1}, \dots, \gamma_n \in L$ such that the (γ_i) form a basis.*
(b) *The matrix $(\gamma_1, \dots, \gamma_m)$, expressing the γ_i in terms of the basis $\boldsymbol{\psi}$, contains a $m \times m$ minor whose determinant is a p-adic unit.*
(c) *If $\sum_{i=1}^n v_i \gamma_i \in L$ for some $v_i \in \mathbb{Q}_\ell$, then $v_i \in \mathbb{Z}_\ell$.*

Proof. — This is straightforward. See also [**C**, Chapter 8, Lemma 2.1]. □

In particular, a vector $\alpha = \sum_i \alpha_i \psi_i \in L$ is part of a basis of L if and only if $\min_j \operatorname{ord}(\alpha_j) = 0$. We call such vectors *primitive*.

We have that

$$2\,(x, y) = 2[Q(x + y) - Q(x) - Q(y)] = (x + y, x + y) - (x, x) - (y, y)\,. \tag{2.1}$$

If $\ell \neq 2$, this implies that

$$\min_{x,y \in L} \operatorname{ord}\,(x, y) = \min_{x \in L} \operatorname{ord}\,(x, x)\,. \tag{2.2}$$

In the rest of this section, we suppose that $\ell \neq 2$. There is a $x \in L$ for which the minimum in (2.2) is attained. This vector x is primitive. Lemma 2.1 implies that x can be extended to a basis of L. We will see in Section 4 that (2.2) does not hold for $\ell = 2$; this is the main reason why things are more difficult for $\ell = 2$.

Proposition 2.2. — *Suppose that $\ell \neq 2$. Then there exists a basis $\boldsymbol{\psi}$ of L such that*

$$Q(x) = Q\Big(\sum x_i \psi_i\Big) = \sum_i b_{ii} x_i^2, \qquad \textit{where} \qquad \operatorname{ord}(b_{11}) \leq \operatorname{ord}(b_{22}) \leq \cdots \leq \operatorname{ord}(b_{nn}).$$

Proof. — Our proof follows [**C**, Chapter 8, Theorem 3.1].

The discussion before the statement of the theorem shows that we may choose φ_1 such that

$$\operatorname{ord}(Q(\varphi_1)) = \operatorname{ord}\,(\varphi_1, \varphi_1) = \min_{x,y \in L} \operatorname{ord}\,(x, y)\,.$$

Here we use the equality (2.2).

Choose $\varphi_2, \dots, \varphi_n \in L$ such that $\boldsymbol{\varphi} = \{\varphi_1, \varphi_2 \dots, \varphi_n\}$ is a basis of L. As before we write $Q(\sum_i x_i \varphi_i) = \sum_{1 \leq i \leq j \leq n} b_{ij}(\boldsymbol{\varphi}) x_i x_j$. Then

$$Q(x) = b_{11}\left(x_1 + \frac{b_{12}}{2b_{11}} x_2 + \cdots \frac{b_{1n}}{2b_{11}} x_n\right)^2 + \tilde{Q}(x_2, \dots, x_n),$$

for some integral quadratic form $\tilde{Q}$ in $n - 1$ variables.

We define a new basis by $\psi_1 = \varphi_1$, and $\psi_i = \varphi_i - (b_{1i}/2b_{11})\varphi_1$ for $i \neq 1$. The choice of ψ_1 ensures that $\psi_i \in L$, since $e = \mathrm{ord}(2b_{11}) \leq \mathrm{ord}(b_{1i})$. With respect to this new basis, the quadratic form is

$$Q(x) = b_{11}(\boldsymbol{\psi})x_1^2 + \tilde{Q}\Bigl(\sum_{i\geq 2} x_i\psi_i\Bigr).$$

The proposition follows by induction. □

Remark 2.3. — Cassels ([**C**, Chapter 8, Theorem 3.1]) proves a stronger statement than Proposition 2.2. Namely, he gives a list of pairwise nonisomorphic quadratic forms such that every integral quadratic form is isomorphic to one of these. This stronger statement implies that the definition of the invariants a_i of Proposition 2.6 does not depend of the choice of the orthogonal basis.

We can give a simpler definition of the invariants a_i in terms of a basis $\boldsymbol{\psi}$ as in Proposition 2.2. If $\gamma \in L$ is an element such that $Q(\gamma) \neq 0$, we may define a reflection τ_γ by

$$\tau_\gamma(x) = x - \frac{2\,(x,\gamma)}{(\gamma,\gamma)}\gamma.$$

This is the reflection in the orthogonal complement of γ. Clearly, τ_γ is defined over $\mathbb{Z}_\ell$ if and only if $\mathrm{ord}\,(\gamma,\gamma) = \min_{x\in L}\mathrm{ord}\,(x,x)$. (In fact, this also holds for $\ell = 2$.) Since τ_γ is a reflection, it is clearly invertible. The following lemma is a partial analog of Witt's Lemma ([**C**, Corollary to Theorem 2.4.1]) which holds for quadratic forms over fields.

Lemma 2.4. — *Suppose that $\psi, \varphi \in L$ satisfy*

$$Q(\psi) = Q(\varphi), \qquad \mathrm{ord}(Q(\psi)) = \mathrm{ord}(Q(\varphi)) = \min_{x\in L}\mathrm{ord}(Q(x)).$$

Then there exists an integral isometry σ of (L,Q) such that $\sigma(\psi) = \varphi$. Moreover, σ may be taken as a product of reflections τ_γ.

Proof. — This is [**C**, Lemma 8.3.3]. Our assumptions on ψ and φ imply that $Q(\psi+\varphi) + Q(\psi-\varphi) = 2Q(\psi) + 2Q(\varphi) = 4Q(\psi)$. Since $\mathrm{ord}(Q(\psi)) = \mathrm{ord}\,(\psi,\psi) = \min_{x\in L}\mathrm{ord}\,(x,x) =: e$, it follows that one of the following holds:

(a) $\mathrm{ord}\,Q(\psi+\varphi) = e$,
(b) $\mathrm{ord}\,Q(\psi-\varphi) = e$.

Since $\ell \neq 2$, it is also possible that both hold. If (a) holds, then $\tau_{\psi+\varphi}$ is integral and sends ψ to φ. If (b) holds, define $\sigma = \tau_{\psi-\varphi}\circ\tau_\psi$. □

Lemma 2.5. — *Suppose $u, v \in \mathbb{Z}_\ell^\times$. Then $ux_1^2 + vx_2^2 \sim_{\mathbb{Z}_\ell} x_1^2 + uvx_2^2$.*