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NEW RESULTS AND PROBLEMS ON KÄHLER-RICCI FLOW

by

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Abstract. — In this paper, I give a brief tour on a program of studying the Kähler-Ricci flow with surgery and its interaction with the classification of projective manifolds. The Kähler-Ricci flow may develops singularity at finite time. It is important to understand how to extend the Kähler-Ricci flow across the singular time, that is, construct solution of the Kähler-Ricci flow with surgery. The first task of this paper is to describe a procedure of constructing global solutions for the Kähler-Ricci flow with surgery. This procedure is rather canonical. I will discuss properties of such solutions with surgery and their geometric implications. I will also discuss their asymptotic limits at time infinity. The results discussed here were mainly from my joint works with Z. Zhang, J. Song *et al.* Some open problems will be also discussed. The paper is mostly expository.

Résumé **(Nouveaux problèmes et résultats sur le flot de Kähler-Ricci).** — Dans cet article, nous donnons un aperçu rapide d'un programme d'études sur le flot de Kähler-Ricci avec chirurgie et son interaction avec la classification des variétés projectives. Le flot de Kähler-Ricci peut développer des singularités en un temps fini. Il est important de comprendre comment étendre le flot de Kähler-Ricci à travers le temps singulier, c'est-à-dire, comment construire une solution du flot de Kähler-Ricci avec chirurgie. La première tâche de cette article consiste à décrire une procédure de construction de solutions globales pour le flot de Kähler-Ricci avec chirurgie. Cette procédure est plutôt canonique. Nous allons discuter les propriétés de telles solutions avec chirurgie et leurs implications géométriques. Nous allons également discuter leurs limites asymptotiques au temps infini. Les résultats présentés ici proviennent principalement de travaux communs avec Z. Zhang, J. Song *et al.* Nous allons également présenter certains problèmes ouverts. L'article est plutôt explicatif.

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1. Introduction

Let X be an n-dimensional compact Kähler manifold. We denote a Kähler metric by its Kähler form ω, in local complex coordinates $z^1, \ldots, z^n$,

$$\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$

where we use the standard convention for summation and $(g_{i\bar{j}})$ is the positive Hermitian matrix valued function given by

$$g_{i\bar{j}} = g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right).$$

The Ricci flow was introduced by R. Hamilton. It has a nice property: If the initial metric is Kählerian, so do any metrics which evolve along the Ricci flow. This can be proved by either using the uniqueness of its local solutions or applying the maximum principle in an appropriate way. Thus we can consider the following Kähler-Ricci flow

$$\frac{\partial \tilde{\omega}_t}{\partial t} = -\mathrm{Ric}(\tilde{\omega}_t), \qquad \tilde{\omega}_0 = \omega_0, \tag{1.1}$$

where ω_0 is any given Kähler metric and $\mathrm{Ric}(\omega)$ denotes the Ricci form of ω, i.e., in the complex coordinates above,

$$\mathrm{Ric}(\omega) = \sqrt{-1} R_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$

where $(R_{i\bar{j}})$ is the Ricci tensor of ω.

This paper is essentially expository. In this paper, I will discuss some new results and open problems in recent study of the Kähler-Ricci flow. They were mainly from my joint works with Z. Zhang, J. Song et al. I will also describe briefly a program of studying the singularity formation of the Kähler-Ricci flow and how it interacts with the classification of projective manifolds. The results and problems discussed here arise from our long efforts in pursuing this program (cf. **[28]**, **[30]**, **[20]**, **[22]**, **[31]**, **[6]** etc.).

2. A sharp local existence for Kähler-Ricci flow

By the local existence of Ricci flow, given any initial Kähler metric ω_0, there is a unique solution $\tilde{\omega}_t$ of (1.1) ($t \in [0, T)$) for some $T > 0$. The following theorem was proved in **[30]** (also see **[2]** (1)) and characterizes the maximal T for which the solution $\tilde{\omega}_t$ exists for $t < T$.

(1) In this cited paper, the authors claimed a proof of a related result under certain extra technical assumptions.

Theorem 2.1. — *Let X be a compact Kähler manifold. Then for any initial Kähler metric ω_0, the flow* (1.1) *has a maximal solution $\tilde{\omega}_t$ on $X \times [0, T_{\max})$, where*

$$T_{\max} = \sup\{t \mid [\omega_0] - t\, c_1(X) > 0^{(2)}\}.$$

In particular, if the canonical class K_X is numerically effective, then (1.1) *has a global solution $\tilde{\omega}_t$ for all $t > 0$. Here, $c_1(X)$ denotes the 2π mutiple of the first Chern class.*

In [**1**], Cao proved this theorem in the case that $c_1(X)$ is definite and proportional to the initial Kähler class. In the case that K_X is nef, i.e., numerically effective, and the initial metric ω_0 is sufficiently positive, H. Tsuji proved in [**32**] the above theorem, that is, (1.1) has a global solution $\tilde{\omega}_t$.

Now let us sketch a proof of the above theorem following the arguments in the proof of Proposition 1.1 in [**30**]. (3)

For any small $\epsilon > 0$, we can choose $T_\epsilon > 0$ such that $T_\epsilon + \epsilon < T_{\max}$ and a real closed $(1,1)$ form ψ_ϵ such that $[\psi_\epsilon] = c_1(X)$ and $\omega_0 - (T_\epsilon + \epsilon)\,\psi_\epsilon \geq 0$. Choose a smooth volume form Ω_ϵ such that $\mathrm{Ric}(\Omega_\epsilon) = \psi_\epsilon$. This Ω_ϵ is unique up to multiplication by a positive constant.

Set $\omega_t = \omega_0 - t\psi_\epsilon$ for $t \in [0, T_\epsilon]$. One can easily show that $\tilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u$ satisfies (1.1) if u satisfies

$$\frac{\partial u}{\partial t} = \log \frac{\tilde{\omega}_t^n}{\Omega_\epsilon}, \qquad u(0,\cdot) = 0. \tag{2.1}$$

We shall show the solution for (2.1) exists for $t \in [0, T_\epsilon]$.

First observe that ω_t is a Kähler metric for $t \in [0, T_\epsilon]$ with uniformly bounded geometry.

By the standard theory, u exists for small $t > 0$. In order to prove that u exists for $t \in [0, T_\epsilon]$, we only need to get uniform estimates of u whenever it exists for $t \in [0, T_\epsilon]$.

Applying the Maximum Principle to (2.1), we can easily have $|u| \leq C_\epsilon$. (4) In fact, the upper bound is independent of ϵ.

Taking derivative of (2.1) with respect to t, we get

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \Delta_{\tilde{\omega}_t}\left(\frac{\partial u}{\partial t}\right) - \langle \tilde{\omega}_t, \psi_\epsilon \rangle,$$

where Δ_ω denotes the Laplacian of a Kähler metric ω and $\langle \omega, F \rangle$ means the trace of F with respect to ω for a real $(1,1)$-form F.

It follows

$$\frac{\partial}{\partial t}\left(t\frac{\partial u}{\partial t} - u\right) = \Delta_{\tilde{\omega}_t}\left(t\frac{\partial u}{\partial t} - u\right) + n - \langle \tilde{\omega}_t, \omega_0 \rangle. \tag{2.2}$$

(2) This means that $[\omega_0] - tc_1(X) > 0$ represents a Kähler class.

(3) The flow equation in [**30**] is not the same as, but equivalent to (1.1).

(4) The constant C, C_ϵ may differ at various places. A subscript indicates the dependence on another constant.

Noticing $\langle\tilde{\omega}_t,\omega_0\rangle > 0$ and applying the Maximum Principle, we see that the maximum of $t\frac{\partial u}{\partial t} - u - nt$ is non-increasing, so we have that

$$t\frac{\partial u}{\partial t} - u - nt \leqslant 0.$$

Now we combine it with local existence for small time and the uniform upper bound for u to conclude that

$$\frac{\partial u}{\partial t} \leq C.$$

On the other hand, we have

$$\begin{aligned}(2.3)\quad & \frac{\partial}{\partial t}\left((T_\epsilon + \epsilon - t)\frac{\partial u}{\partial t} + u\right)\\ &= \Delta_{\tilde{\omega}_t}\left((T_\epsilon + \epsilon - t)\frac{\partial u}{\partial t} + u\right) - n + \langle\tilde{\omega}_t, \omega_0 - (T_\epsilon + \epsilon)\psi_\epsilon\rangle.\end{aligned}$$

Since $\langle\tilde{\omega}_t, \omega_0 - (T_\epsilon + \epsilon)\psi_\epsilon\rangle \geq 0$, by the Maximum Principle, we see that minimum of $(T_\epsilon + \epsilon - t)\frac{\partial u}{\partial t} + u + nt$ is non-decreasing. It follows

$$(T_\epsilon + \epsilon - t)\frac{\partial u}{\partial t} + u + nt \geqslant (T_\epsilon + \epsilon)\min_{t=0}\frac{\partial u}{\partial t} = -C_\epsilon,$$

from this we can conclude

$$\frac{\partial u}{\partial t} > -C_\epsilon.$$

Now we have gotten all the C^0-estimates needed. By using the Maximum principle and the standard arguments, one can derive the second and higher order estimates for u (cf. [**30**] for more details). Then one obtains the existence of solution for (2.1) for $t \in [0, T_\epsilon]$.

The desired existence of the solution for (1.1) can be proved by considering the relations between all the equations as (2.1) for different ϵ's as follows:

Consider (2.1) for some $\delta > 0$. Assume $\psi_\delta = \psi_\epsilon + \sqrt{-1}\partial\bar{\partial}f$ for some smooth real function f over X. Since $\mathrm{Ric}_{\Omega_\epsilon} = \psi_\epsilon$, we have $\mathrm{Ric}_{e^{-f}\Omega_\epsilon} = \psi_\delta$. Thus we can take $\Omega_\delta = e^{-f}\Omega_\epsilon$. Now the new "$\omega_t$" is

$$\eta_t = \omega_0 - t\psi_\delta = \omega_t - t\sqrt{-1}\partial\bar{\partial}f.$$

The equation (2.1) for δ is

$$\frac{\partial v}{\partial t} = \log\frac{(\eta_t + \sqrt{-1}\partial\bar{\partial}v)^n}{e^f\Omega_\epsilon}, \qquad v(0,\cdot) = 0.$$

Define $\tilde{u} = v - tf$. Then

$$\begin{aligned}(2.4)\quad \frac{\partial \tilde{u}}{\partial t} = \frac{\partial v}{\partial t} - f &= \log\frac{(\eta_t + \sqrt{-1}\partial\bar{\partial}v)^n}{e^{-f}\Omega_\epsilon} + f\\ &= \log\frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\tilde{u})^n}{\Omega_\epsilon}.\end{aligned}$$