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DRIFT OF RANDOM WALKS ON ABELIAN COVERS OF FINITE VOLUME HOMOGENEOUS SPACES

by Timothée Bénard

Abstract. — Let G be a connected simple real Lie group, $\Lambda_0 \subseteq G$ a lattice without torsion and $\Lambda \trianglelefteq \Lambda_0$ a normal subgroup such that $\Lambda_0/\Lambda \simeq \mathbb{Z}^d$. We study the drift of a random walk on the $\mathbb{Z}^d$-cover $\Lambda \setminus G$ of the finite volume homogeneous space $\Lambda_0 \setminus G$. This walk is defined by a Zariski-dense compactly supported probability measure μ on G. We first assume the covering map $\Lambda\setminus G \to \Lambda_0\setminus G$ does not unfold any cusp of $\Lambda_0\setminus G$ and compute the drift at *every* starting point. Then we remove this assumption and describe the drift almost everywhere. The case of hyperbolic manifolds of dimension 2 stands out with non-converging type behaviors. The recurrence of the trajectories is also characterized in this context.

Résumé (*Dérive d'une marche aléatoire sur un revêtement abélien d'un espace homogène de volume fini*). — Soit G un groupe de Lie réel connexe, $\Lambda_0 \subseteq G$ un réseau sans torsion dans G, et $\Lambda \trianglelefteq \Lambda_0$ un sous-groupe distingué tel que $\Lambda_0/\Lambda \simeq \mathbb{Z}^d$. Nous étudions la dérive d'une marche aléatoire sur le $\mathbb{Z}^d$-revêtement $\Lambda \setminus G$ de l'espace homogène de volume fini $\Lambda_0 \setminus G$. Cette marche est définie par une mesure de probabilité Zariski-dense à support compact μ sur G. Nous supposons dans un premier temps que l'application de revêtement $\Lambda \setminus G \to \Lambda_0 \setminus G$ ne déroule pas les pointes de $\Lambda_0 \setminus G$ et calculons la dérive en *tout* point de départ. Ensuite, nous travaillons sans cette hypothèse et décrivons la dérive en presque tout point. Le cas des variétés hyperboliques de dimension 2 se démarque par des comportements non convergents. La récurrence presque-sûre des trajectoires est aussi caractérisée dans ce contexte.

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Introduction

$\mathbb{Z}^d$-covers of finite volume hyperbolic surfaces are important examples to study dynamical systems in infinite measure. They have the advantage of being very concrete, allow the use of tools developed for finite volume spaces via the periodicity of the covering but still present a true complexity, which sometimes leads to unexpected developments.

Such covers have been extensively studied in terms of their geodesic flow or horocycle flow. Let us mention the work of Le Jan, Guivarc'h, Enriquez [16, 14, 17, 12], which reveal that the distribution of a fundamental domain spreading under the action of the geodesic flow obeys a central limit theorem (partially Gaussian, partially Cauchy). On a similar subject, Oh and Pan obtained a local limit theorem in [19, 20]. Concerning the horocycle flow, spectacular progress has been achieved by Sarig in [22], adding the final piece to the classification of Radon invariant measures initiated by Babillot–Ledrappier [3, 2], and realizing a first step toward an extension of Ratner's theorems for infinite volume homogeneous spaces.

The present paper adopts a different point of view, looking into the dynamics of random walks on $\mathbb{Z}^d$-covers of finite volume homogeneous spaces. Even if the dynamics is no longer deterministic, it shares strong relations with the works cited above, which will be key ingredients in several proofs below. Recent investigations have been carried out in this context by Conze and Guivarc'h, who in [11] described the recurrence properties of symmetric random walks when the base of the cover is compact. Our purpose is to examine the rate of escape of a walk trajectory at linear scale, in the spirit of a law of large numbers. Loosely speaking, we will show that the drift of a (not necessarily symmetric) walk is always null, except when the base of the cover is a hyperbolic surface with unfolded cusps, in which case, non-converging behaviors occur.

The homogeneous spaces that we consider are given by quotients of quasi-simple Lie groups. We denote by G a connected real Lie group with simple Lie algebra and finite center, $\Lambda_0 \subseteq G$ a lattice without torsion, $\Lambda \trianglelefteq \Lambda_0$ a normal subgroup such that $\Lambda_0/\Lambda \simeq \mathbb{Z}^d$, and look into the quotient space $X = \Lambda \setminus G$, which is a $\mathbb{Z}^d$-cover of the finite volume homogeneous space $X_0 = \Lambda_0 \setminus G$. This context is actually very explicit, as we can assume without loss of generality that $G = SO_e(1, m)$ or $G = SU(1, m)$ for some integer $m \geq 2$ (see 1.1). In this case, X_0 corresponds to the orthonormal frame bundle of a real or complex hyperbolic manifold of finite volume M_0, and X to the orthonormal frame bundle of some $\mathbb{Z}^d$-cover M of M_0.

To define a random walk on X or X_0, we choose a probability measure μ on G and define the transitional probability measure at a point x as the convolution $\delta_x \star \mu$, i.e., as the image of μ by the map $g \mapsto xg$. Set $B = G^{\mathbb{N}^\star}$, $\beta = \mu^{\otimes \mathbb{N}^\star}$. A typical trajectory for the μ-walk starting from a point $x \in X$ is thus obtained by choosing an element $b \in B$ with law β and considering the sequence $(xb_1 \dots b_n)_{n\geq 0} \in X^{\mathbb{N}}$. In the following, we will always assume that the support of the measure μ is *compact* and generates a *Zariski-dense* sub-semigroup Γ_μ in G.

To express the rate of escape of the μ-walk on X, we fix a fundamental domain D for the action of $\mathbb{Z}^d$ on X such that D lifts well the cusps of X_0 (see 1.3) and some $\mathbb{Z}^d$-equivariant measurable map $i : X \to \mathbb{R}^d$ that is bounded on D. The role of this map is to associate to every point $x \in X$ some coordinates $i(x)$ in $\mathbb{R}^d$ that quantify which $\mathbb{Z}^d$-translate of D contains x.

The law of large numbers now inspires the following questions. Let $(xb_1 \dots b_n)_{n\geq 0}$ be a typical trajectory of the μ-walk on X.

Does the sequence $\frac{1}{n}i(xb_1 \dots b_n)_{n\geq 1}$ converge in $\mathbb{R}^d$?
And if so, what is the limit?

To discuss this issue, we first clarify the notion of *drift*. Let $x \in X$ be a point on X and E a subset of $\mathbb{R}^d$. We say the μ-walk with starting point x has drift E if for β-almost every $b \in B$, the set of accumulation points of the normalized sequence of positions $\left(\frac{1}{n}i(xb_1 \dots b_n)\right)_{n\geq 1}$ is equal to E. When E is a singleton, we recover the more common definition of drift, as it appears in the classical law of large numbers. However, we will see examples where E is not reduced to a point.

The drift of the μ-walk on X is linked with the dynamical properties of a cocycle on $X_0 \times G$ that we now introduce. Notice first that each step of the walk yields a variation of the index of position given by the cocycle

$$X \times G \to \mathbb{R}^d, \quad (x, g) \mapsto i(xg) - i(x)$$

The assumption that i commutes with the action of $\mathbb{Z}^d$ implies that this cocycle is $\mathbb{Z}^d$-invariant and, hence, defines a quotient *drift cocycle*:

$$\sigma : X_0 \times G \to \mathbb{R}^d, \quad (x + \mathbb{Z}^d, g) \mapsto i(xg) - i(x).$$

As for $x \in X$, $b \in B$, one has $i(xb_1 \dots b_n) = \sigma(x + \mathbb{Z}^d, b_1 \dots b_n) + i(x)$; the drift at x is also the set of accumulation points of β-typical sequences $\left(\frac{1}{n}\sigma(x + \mathbb{Z}^d, b_1 \dots b_n)\right)_{n\geq 1}$. Hence, we will freely talk about the drift of the μ-walk at a point that is not on X but on X_0.

Now, the ergodicity[1] of the μ-walk on X_0 with respect to the Haar probability measure λ_0 gives a first answer to our questions: *if σ is $\lambda_0 \otimes \mu$-integrable*, then by the Birkhoff ergodic theorem, for λ_0-almost every $x_0 \in X_0$, β-almost every $b \in B$,

$$\frac{1}{n}\sigma(x_0, b_1 \dots b_n) \xrightarrow[n\to+\infty]{} \int_{X_0\times G} \sigma(x, g)\, d\lambda_0(x) d\mu(g).$$

1. It is a consequence of [10, Proposition 2.8], the strict convexity of balls in a Hilbert space, and the Howe–Moore theorem [24, Theorem 2.2.15].