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LOCAL VANISHING MEAN OSCILLATION

by Almaz Butaev & Galia Dafni

Abstract. — We consider various notions of vanishing mean oscillation on a (possibly unbounded) domain $\Omega \subset \mathbb{R}^n$, and prove an analogue of Sarason's theorem, giving sufficient conditions for the density of bounded Lipschitz functions in the nonhomogeneous space $\mathrm{vmo}(\Omega)$. We also study $\mathrm{cmo}(\Omega)$, the closure in $\mathrm{bmo}(\Omega)$ of the continuous functions with compact support in Ω. Using these approximation results, we prove that there is a bounded extension from $\mathrm{vmo}(\Omega)$ and $\mathrm{cmo}(\Omega)$ to the corresponding spaces on $\mathbb{R}^n$, if and only if Ω is a locally uniform domain.

Résumé (*VMO locale*). — Plusieurs notions de *vanishing mean oscillation* pour des fonctions définies sur des domaines $\Omega \subset \mathbb{R}^n$ sont abordées et une version du théorème de Sarason qui donne une condition suffisante pour la densité dans l'espace non homogène $\mathrm{vmo}(\Omega)$ des fonctions lipschitziennes bornées est démontrée. De plus, l'espace $\mathrm{cmo}(\Omega)$, l'adhérence de l'ensemble des fonctions continues à support compact dans $\mathrm{bmo}(\Omega)$, est caractérisé. Ces résultats d'approximation nous permettent de montrer qu'un opérateur de prolongement borné de $\mathrm{vmo}(\Omega)$ et $\mathrm{cmo}(\Omega)$ vers les espaces correspondants sur $\mathbb{R}^n$ existe si et seulement si Ω est un domaine localement uniforme.

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1. Introduction

In the theory of function spaces, it is useful to define "local" versions of certain spaces. For function spaces on $\mathbb{R}^n$, this can refer to functions defined globally, which belong to a certain space when restricted to compact sets (such as $L^p_{\rm loc}(\mathbb{R}^n)$) or alternatively to spaces defined on a given domain $\Omega \subset \mathbb{R}^n$. In the latter case, one can then use "local" to refer to behavior away from the boundary $\partial\Omega$. When considering behavior up to the boundary, the notions of "vanishing" at the boundary, approximation by smooth functions, and extension to $\mathbb{R}^n$ are interrelated and intimately connected with the geometry of the domain (for example, in the case of Sobolev spaces and Triebel–Lizorkin spaces, see [5, 16, 17, 15, 20, 23]).

In the case of functions of bounded mean oscillation, there are various notions of "local" and "vanishing" in the literature. The original definition of bounded mean oscillation by John and Nirenberg [13] was on a fixed cube $Q_0 \subset \mathbb{R}^n$: $f \in \mathrm{BMO}(Q_0)$ if

$$\sup_{Q\subset Q_0} \fint_Q |f(x) - f_Q|dx < \infty,$$

where the supremum is over all parallel subcubes of Q_0, $|Q|$ denotes Lebesgue measure, and $f_Q := \fint_Q f := |Q|^{-1}\int_Q f$ is the average of f on Q. This can be extended to give a definition of $\mathrm{BMO}(\Omega)$ on any domain (or even open set) $\Omega \subset \mathbb{R}^n$ by taking the supremum over all cubes $Q \subset \Omega$ with sides parallel to the axes. When Ω is connected, this supremum defines a norm modulo constants, and $\mathrm{BMO}(\Omega)$ is a Banach space.

A more refined measure of the mean oscillation of a function $f \in L^1_{\rm loc}(\mathbb{R}^n)$ is given by the *modulus of mean oscillation*

$$\omega(f,t) := \sup_{\substack{\ell(Q)<t \\ Q\subset\mathbb{R}^n}} \fint_Q |f(x) - f_Q|dx, \quad t > 0, \tag{1}$$

where $\ell(Q)$ denotes the sidelength of Q. We then have $\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_{t>0}\omega(f,t)$. A local version of BMO can be defined by fixing a finite $T > 0$ and considering locally integrable functions with $\omega(f,T) < \infty$. The set of such functions does not depend on the choice of T and is strictly larger than $\mathrm{BMO}(\mathbb{R}^n)$ since it contains, for example, all uniformly continuous functions.

Another version of BMO that is called "local" is the space $\mathrm{bmo}(\mathbb{R}^n)$ (not to be confused with what is known as "little" BMO and has the same notation), introduced by Goldberg [12] as the dual of the local Hardy space $h^1(\mathbb{R}^n)$ and consisting of locally integrable functions f satisfying

$$\|f\|_{\mathrm{bmo}(\mathbb{R}^n)} := \omega(f,1) + \sup_{\ell(Q)\geq 1} |f|_Q < \infty,$$

where once more the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the axes. Here again the scale 1 can be replaced by any finite T without changing the collection of functions in $\mathrm{bmo}(\mathbb{R}^n)$, only affecting the norm, and we can also restrict the second term to the supremum of the averages of $|f|$ on cubes whose sidelength is exactly equal to 1. As sets of functions, $\mathrm{bmo}(\mathbb{R}^n)$ is strictly smaller than $\mathrm{BMO}(\mathbb{R}^n)$ (it does not contain $\log|x|$, for example) and should be considered as a nonhomogeneous version of BMO, not taken modulo constants. Both h^1 and bmo are part of the scale of nonhomogeneous Triebel–Lizorkin spaces – see [28, Theorem 1.7.1].

The notion of vanishing mean oscillation was introduced by Sarason [21]. The space $\mathrm{VMO}(\mathbb{R}^n)$ can be defined using either one of the two characterizations in the following theorem, which was proved in [21] for the case $n = 1$.

Theorem A (Sarason). — *For* $f \in \mathrm{BMO}(\mathbb{R}^n)$,

$$\lim_{t\to 0^+} \omega(f,t) = 0 \tag{2}$$

if and only if $f \in \overline{\mathrm{UC}(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)}$, *the closure of the uniformly continuous functions in* BMO.

A smaller space, which is sometimes also called $\mathrm{VMO}(\mathbb{R}^n)$ (see [10]) and serves as a predual to the Hardy space $H^1(\mathbb{R}^n)$, is the closure in $\mathrm{BMO}(\mathbb{R}^n)$ of the continuous functions with compact support (or equivalently the C^∞ functions with compact support). We will denote this space by $\mathrm{CMO}(\mathbb{R}^n)$ for "continuous mean oscillation", following Neri [19]. As stated in [19] and proved by Uchiyama in [29], in addition to (2), functions in $\mathrm{CMO}(\mathbb{R}^n)$ also satisfy vanishing mean oscillation conditions as the size of the cube increases to ∞ and as the cube itself goes to ∞. Recently, subspaces between $\mathrm{CMO}(\mathbb{R}^n)$ and $\mathrm{VMO}(\mathbb{R}^n)$ were considered in [26, 27].

The nonhomogeneous versions of the spaces $\mathrm{VMO}(\mathbb{R}^n)$ and $\mathrm{CMO}(\mathbb{R}^n)$, denoted $\mathrm{vmo}(\mathbb{R}^n)$ and $\mathrm{cmo}(\mathbb{R}^n)$, are the corresponding subspaces of $\mathrm{bmo}(\mathbb{R}^n)$, and the vanishing mean oscillation conditions characterizing the latter were given in [2, 11] (see Proposition 2.1). Bourdaud's paper [2] contains extensive coverage of the properties of $\mathrm{BMO}(\mathbb{R}^n)$ and $\mathrm{bmo}(\mathbb{R}^n)$ (treating it modulo constants as a subspace of BMO), as well as their vanishing subspaces.

The focus of our work is on versions of these spaces on a domain $\Omega \subset \mathbb{R}^n$, and the corresponding approximation and extension results. The definition of the modulus of oscillation can be adapted by restricting the cubes to lie inside the domain, namely

$$\omega_\Omega(f,t) := \sup_{\substack{\ell(Q)<t \\ Q\subset\Omega}} \fint_Q |f(x) - f_Q|dx, \quad t > 0, \tag{3}$$

and $\mathrm{BMO}(\Omega)$ defined to consist of those $f \in L^1_{\mathrm{loc}}(\Omega)$ with $\sup_{t>0} \omega_\Omega(f,t) < \infty$. The question of the definition of $\mathrm{VMO}(\Omega)$ is more delicate: for which domains