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CENTRAL EXTENSIONS OF PREORDERED GROUPS

by Marino Gran & Aline Michel

Abstract. — We prove that the category of preordered groups contains two full reflective subcategories that give rise to some interesting Galois theories. The first one is the category of so-called commutative objects, which are precisely the preordered groups whose group law is commutative. The second one is the category of abelian objects, which turns out to be the category of monomorphisms in the category of abelian groups. We give a precise description of the reflector to this subcategory and we prove that it induces an admissible Galois structure and then a natural notion of *categorical central extension*. We then characterize the central extensions of preordered groups in purely algebraic terms; these are shown to be the central extensions of groups having the additional property that their restriction to positive cones is a special Schreier surjection of monoids.

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Résumé (*Extensions centrales de groupes préordonnés*). — Nous prouvons que la catégorie des groupes préordonnés contient deux sous-catégories pleines réflexives qui donnent lieu à certaines théories de Galois intéressantes. La première est la catégorie des objets commutatifs, qui sont précisément les groupes préordonnés dont la loi de groupe est commutative. La seconde est la catégorie des objets abéliens, qui s'avère être la catégorie des monomorphismes dans la catégorie des groupes abéliens. Nous donnons une description précise du réflecteur vers cette sous-catégorie, et nous prouvons qu'il induit une structure galoisienne admissible et donc une notion naturelle d'extension centrale catégorique. Nous caractérisons ensuite les extensions centrales de groupes préordonnés en termes purement algébriques : on montre qu'elles sont données par les extensions centrales de groupes ayant la propriété additionnelle que leur restriction aux cônes positifs est une surjection spéciale de Schreier de monoïdes.

Introduction

A *preordered group* $(G, \leq)$ is a group $G = (G, +, 0)$ endowed with a preorder relation $\leq$ that is compatible with the addition $+$ of the group G, in the sense that, if $a \leq c$ and $b \leq d$, then $a+b \leq c+d$ (for $a, b, c, d \in G$). Preordered groups and monotone group homomorphisms form a category, denoted by $\mathsf{PreOrdGrp}$. This category is actually isomorphic to another category, whose objects are given by pairs (G, P_G), where G is a group and P_G a submonoid of G closed under conjugation in G. This submonoid P_G is usually called the *positive cone* of G, and an object (G, P_G) can be depicted as

$$P_G \rightarrowtail G,$$

where the arrow represents the inclusion of P_G in G. An arrow between two such objects (G, P_G) and (H, P_H) is given by a pair $(f, \bar{f})\colon (G, P_G) \to (H, P_H)$ of monoid morphisms making the diagram

$$\begin{array}{ccc} P_G & \xrightarrow{\bar{f}} & P_H \\ \downarrow & & \downarrow \\ G & \xrightarrow[f]{} & H \end{array} \tag{1}$$

commute, so that $f\colon G \to H$ is a group homomorphism that "restricts" to the positive cones, in the sense that $f(P_G) \subseteq P_H$. It is this second equivalent definition of the category $\mathsf{PreOrdGrp}$ of preordered groups that we shall use throughout this article. As shown in [4] the category $\mathsf{PreOrdGrp}$ is both complete and cocomplete and is a *normal* category in the sense of [14], which is a pointed regular category where every regular epimorphism is a normal epimorphism (i.e. a cokernel).

In this article, we prove that the lattice of normal subobjects on any preordered group (G, P_G) is *modular* (Proposition 2.5), and this implies that any reflective subcategory of PreOrdGrp that is also closed in it under subobjects and regular quotients is *admissible* from the point of view of categorical Galois theory [7] (see Proposition 2.3). In particular, the full subcategory PreOrdAb of preordered *abelian* groups satisfies this property, giving rise to the adjunction

$$\mathsf{PreOrdGrp} \underset{V}{\overset{C}{\rightleftarrows}} \mathsf{PreOrdAb}, \qquad C \dashv V \tag{2}$$

where V is the inclusion functor, and its left adjoint C sends a preordered group (G, P_G) to the preordered abelian group $(G/[G,G], \eta_G(P_G))$, where $\eta_G : G \twoheadrightarrow G/[G,G]$ is the quotient of G by its derived subgroup $[G,G]$. Preordered abelian groups turn out to be precisely the *commutative objects* (in the sense of [2]) of the category PreOrdGrp. A characterization of the normal extensions of preordered groups with respect to this adjunction is given in Theorem 2.9; these are precisely the normal epimorphisms $(f, \bar{f}) : (G, P_G) \longrightarrow\!\!\!\!\rightarrow (H, P_H)$ such that $f : G \longrightarrow\!\!\!\!\rightarrow H$ is a central extension of groups and, moreover, $a - b + c \in P_G$ whenever $a, b, c \in P_G$ are such that $\eta_G(a) = \eta_G(b)$ and $f(b) = f(c)$.

We then turn our attention to the composite adjunction

$$\mathsf{PreOrdGrp} \underset{V}{\overset{C}{\rightleftarrows}} \mathsf{PreOrdAb} \underset{W}{\overset{A}{\rightleftarrows}} \mathsf{Mono(Ab)}, \qquad C \dashv V,\ A \dashv W$$

where Mono(Ab) is the category of monomorphisms in the category Ab of abelian groups, W is the inclusion functor and A its left adjoint (described in detail in Section 3). We prove that Mono(Ab) is the category of *abelian objects* in PreOrdGrp (Corollary 3.8), and we characterize the normal epimorphisms

$$\begin{array}{ccc} P_G & \overset{\bar{f}}{\twoheadrightarrow} & P_H \\ \downarrow & & \downarrow \\ G & \underset{f}{\twoheadrightarrow} & H \end{array} \tag{3}$$

in PreOrdGrp (i.e. both f and $\bar{f}$ are surjective) that are *central extensions* in the sense of categorical Galois theory [10] for this composite adjunction. By using the results established in [15] we show in Theorem 6.3 that these are characterized by the fact that the surjective morphism f is a central extension of groups and $\bar{f}$ a special homogeneous surjection (or, equivalently, a special Schreier surjection) in the sense of [3]. This result opens the way to the possibility of studying the non-abelian homology of preordered groups by using the approach adopted in [5] (which is itself based on the one in [9]), which we leave for future work.