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DIOPHANTINE APPROXIMATION WITH WEIGHTS AND TITS BUILDINGS AT INFINITY OF SYMMETRIC SPACES

BY TOSHIAKI HATTORI

ABSTRACT. — Let m, n be positive integers and $M(m, n; \mathbf{R})$ the set of all $m \times n$ real matrices. For any given n-dimensional weight and m-dimensional weight, we associate each $L \in M(m, n; \mathbf{R})$ with a geodesic ray in the symmetric spaces $M = SL(m+n, \mathbf{R})/SO(m+n)$ and consider its "end point" in the geometric boundary $M(\infty)$ of M. We study Diophantine properties of the system of linear forms induced from L by using the structures of Tits buildings on $M(\infty)$.

RÉSUMÉ (*Approximation diophantienne avec poids et immeubles de Tits à l'infini d'espaces symétriques*). — Soient m, n des entiers positifs et $M(m, n; \mathbf{R})$ l'ensemble de toutes $m \times n$ matrices réelles. Nous associons à chaque $L \in M(m, n; \mathbf{R})$ un rayon géodésique dans l'espace symétrique $M = Sl(m+n, \mathbf{R})/SO(m+n)$, pour des poids donnés, et considérons son "extrémité" dans la frontière géométrique $M(\infty)$ de M. Nous étudions les propriétés diophantiennes du système de formes linéaires induit par L en utilisant les structures des immeubles de Tits sur $M(\infty)$.

1. Introduction

Let m, n be positive integers and $M(m, n; \mathbf{R})$ the set of all $m \times n$ real matrices. For any given n-dimensional weight $\mathbf{r}$ and m-dimensional weight $\mathbf{s}$, one can associate each $L \in M(m, n; \mathbf{R})$ with a trajectory in the quotient

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space $SL(m+n,\mathbf{Z})\backslash SL(m+n,\mathbf{R})$ in a natural way, and the behavior of this trajectory reflects some Diophantine properties of the system of linear forms induced from the matrix L. This is a weighted generalization of the Dani correspondence ([13]) shown by Kleinbock in [26]. On the other hand, it is possible to associate L with a geodesic ray $\gamma_{L;\mathbf{r},\mathbf{s}}$ in the symmetric space $M = SL(m+n,\mathbf{R})/SO(m+n)$ in a similar way. Then its behavior also reflects properties of the system of linear forms. In this case, we can also associate L with the "end point" of $\gamma_{L;\mathbf{r},\mathbf{s}}$ in the geometric boundary $M(\infty)$ of M, which has the structure of a Tits building. In this paper, we study weighted approximation by considering the locations of the end points of such geodesic rays in $M(\infty)$.

1.1. Diophantine approximation with weights. — For $L = (l_{ij}) \in M(m,n;\mathbf{R})$ and $j = 1,\dots,n$, let L_j be the linear form induced from the jth column of L:

$$L_j(\mathbf{x}) = \sum_{k=1}^{m} l_{kj}x_k \quad \text{for } \mathbf{x} = (x_1,\dots,x_m) \in \mathbf{R}^m.$$

We write

$$\mathbf{L}(\mathbf{x}) = (L_1(\mathbf{x}),\dots,L_n(\mathbf{x})) \quad \text{for } \mathbf{x} \in \mathbf{R}^m.$$

For any positive integer k, let

$$W(k) = \left\{\mathbf{r} = (r_1,\dots,r_k) \in \mathbf{R}^k \mid r_1,\dots,r_k > 0;\ r_1+\cdots+r_k = 1\right\}$$

be the set of k-dimensional weights. For each $\mathbf{r} = (r_1,\dots,r_k) \in W(k)$ we define the $\mathbf{r}$-quasinorm $\|\cdot\|_{\mathbf{r}}$ on $\mathbf{R}^k$ (in the sense of [26, Section 2.1]) by

$$\|\mathbf{x}\|_{\mathbf{r}} = \max_{1\le i\le k} |x_i|^{1/r_i} \quad \text{for } \mathbf{x} = (x_1,\dots,x_k) \in \mathbf{R}^k.$$

Let

$$\mathbf{r} = (r_1,\dots,r_n) \in W(n), \quad \mathbf{s} = (s_1,\dots,s_m) \in W(m).$$

Then the system of linear forms $L_1,\dots,L_n$, or the matrix L is said to be $(\mathbf{r},\mathbf{s})$-badly approximable if there exists a positive constant C such that

$$\|\mathbf{x}\|_{\mathbf{s}}\|\mathbf{L}(\mathbf{x}) - \mathbf{y}\|_{\mathbf{r}} \ge C \quad \text{for any } \mathbf{x} \in \mathbf{Z}^m \setminus \{\mathbf{0}\},\ \mathbf{y} \in \mathbf{Z}^n.$$

We say that the system of linear forms $L_1,\dots,L_n$, or the matrix L is $(\mathbf{r},\mathbf{s})$-singular if, for each $\varepsilon > 0$, the set of inequalities

$$\|\mathbf{L}(\mathbf{x}) - \mathbf{y}\|_{\mathbf{r}} < \varepsilon C^{-1}, \qquad \|\mathbf{x}\|_{\mathbf{s}} \le C$$

has an integral solution $(\mathbf{x},\mathbf{y}) \in \mathbf{Z}^m \times \mathbf{Z}^n$ with $\mathbf{x} \ne \mathbf{0}$ for all C greater than some positive number $C_0(\varepsilon)$.

For any positive integer k, let $\|\cdot\|$ be the norm on $\mathbf{R}^k$ defined by

$$\|\mathbf{x}\| = \max_{1\le i\le k} |x_i| \quad \text{for } \mathbf{x} = (x_1,\dots,x_k) \in \mathbf{R}^k.$$

If $\mathbf{r} = (1/n, \dots, 1/n)$ and $\mathbf{s} = (1/m, \dots, 1/m)$, then we have

$$\|\mathbf{x}\|_{\mathbf{r}} = (\|\mathbf{x}\|)^n, \ \ \|\mathbf{y}\|_{\mathbf{s}} = (\|\mathbf{y}\|)^m \quad \text{for } \mathbf{x} = (x_1, \dots, x_n), \ \mathbf{y} = (y_1, \dots, y_m).$$

Hence $(\mathbf{r}, \mathbf{s})$-badly approximable (respectively $(\mathbf{r}, \mathbf{s})$-singular) systems or matrices are badly approximable (respectively singular) systems or matrices in this case.

For $\mathbf{r} \in W(n)$ and $\mathbf{s} \in W(m)$, let $\mathrm{Bad}(m, n; \mathbf{r}, \mathbf{s})$ be the set of matrices $L \in M(m, n; \mathbf{R})$ such that the system of linear forms induced from L is $(\mathbf{r}, \mathbf{s})$-badly approximable. In particular, we write $\mathrm{Bad}(m, n)$ instead of $\mathrm{Bad}(m, n; \mathbf{r}, \mathbf{s})$ if $\mathbf{r} = (1/n, \dots, 1/n)$ and $\mathbf{s} = (1/m, \dots, 1/m)$. In the case $m = 1$, we write $\mathrm{Bad}(1, n; \mathbf{r})$ instead of $\mathrm{Bad}(1, n; \mathbf{r}, \mathbf{s})$.

For $\mathbf{r} \in W(n)$ and $\mathbf{s} \in W(m)$, let $\mathrm{Sing}(m, n; \mathbf{r}, \mathbf{s})$ be the set of matrices $L \in M(m, n; \mathbf{R})$ such that the system of linear forms induced from L is $(\mathbf{r}, \mathbf{s})$-singular. In particular, we write $\mathrm{Sing}(m, n)$ instead of $\mathrm{Sing}(m, n; \mathbf{r}, \mathbf{s})$ if $\mathbf{r} = (1/n, \dots, 1/n)$ and $\mathbf{s} = (1/m, \dots, 1/m)$. In the case $m = 1$, we write $\mathrm{Sing}(1, n; \mathbf{r})$ instead of $\mathrm{Sing}(1, n; \mathbf{r}, \mathbf{s})$.

For any subset A of $M(m, n; \mathbf{R})$, let $\dim_{\mathrm{H}}(A)$ denote the Hausdorff dimension of A under the natural identification $M(m, n; \mathbf{R}) = \mathbf{R}^{mn}$.

1.2. Weighted badly approximable matrices. — It is known that

$$\dim_{\mathrm{H}}(\mathrm{Bad}(m, n; \mathbf{r}, \mathbf{s})) = mn$$

and, in particular, $\mathrm{Bad}(m, n; \mathbf{r}, \mathbf{s})$ is thick for any $\mathbf{r} \in W(n)$ and $\mathbf{s} \in W(m)$ ([24], [33], [30], [28]).

In the case $m = 1$, the sets $\mathrm{Bad}(1, n; \mathbf{r})$ have been studied extensively by many authors in connection with Schmidt's conjecture in [35]. Schmidt conjectured, in the case $n = 2$, that $\mathrm{Bad}(1, 2; \mathbf{r}) \cap \mathrm{Bad}(1, 2; \mathbf{r}') \neq \emptyset$ for any $\mathbf{r}, \mathbf{r}' \in W(2)$. This was proved by Badziahin, Pollington and Velani ([3]), and An ([1]) also showed that $\bigcap_{k=1}^{\infty} \mathrm{Bad}(1, 2; \mathbf{r}_k)$ is thick for any countable sequence $\{\mathbf{r}_k\}_{k=1}^{\infty} \subset W(2)$. Recently, it was shown by Beresnevish, Nesharim and Yang ([7]) that $\bigcap_{k=1}^{\infty} \mathrm{Bad}(1, n; \mathbf{r}_k)$ is thick for any countable sequence $\{\mathbf{r}_k\}_{k=1}^{\infty} \subset W(n)$ for general n.

Compared with the case $m = 1$, not much is known concerning the case $m \geq 2$.

In general, it is natural to expect that

$$\mathrm{Bad}(m, n; \mathbf{r}, \mathbf{s}) \cap \mathrm{Bad}(m, n; \mathbf{r}', \mathbf{s}') \subsetneq \mathrm{Bad}(m, n; \mathbf{r}, \mathbf{s}) \tag{1.1}$$

if $(\mathbf{r}, \mathbf{s}) \neq (\mathbf{r}', \mathbf{s}')$. However, it seems that there are no known concrete examples showing (1.1), at least for the case $m, n \geq 2$. We find such examples as follows.

Let $\mathfrak{S}_m$ be the group of all permutations of m letters and let $M(m, m; \mathbf{Q})$ be the set of all $m \times m$ matrices with coefficients in $\mathbf{Q}$. For any $\sigma \in \mathfrak{S}_m$, let

$$\mathcal{R}_\sigma = \left\{ Z = (z_{ij}) \in M(m, m; \mathbf{Q}) \,|\, z_{i\,\sigma(i)} = 0 \text{ for } i = 1, \dots, m \right\},$$