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Benoît Joly

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Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
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`bulletin@smf.emath.fr` • `smf.emath.fr`

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THE CALABI INVARIANT FOR HAMILTONIAN DIFFEOMORPHISMS OF THE UNIT DISK

BY BENOÎT JOLY

ABSTRACT. — In this article, we study the Calabi invariant on the unit disk usually defined on compactly supported Hamiltonian diffeomorphisms of the open disk. In particular, we extend the Calabi invariant to the group of C^1 diffeomorphisms of the closed disk, which preserves the standard symplectic form. We also compute the Calabi invariant for some diffeomorphisms of the disk that satisfy some rigidity hypothesis.

RÉSUMÉ (*L'invariant de Calabi des difféomorphismes hamiltoniens du disque unité*). — Dans cet article, nous étudions l'invariant de Calabi du disque unité, généralement défini pour les difféomorphismes hamiltoniens à support compact du disque unité ouvert. En particulier, nous étendons l'invariant de Calabi au groupe des difféomorphismes C^1 du disque unité fermé préservant la forme symplectique standard. Nous calculons également l'invariant de Calabi de certains difféomorphismes satisfaisant certaines hypothèses de rigidité.

1. Introduction

Let us begin with some basic definitions of symplectic geometry.

Let us consider (M^{2n}, ω) a *symplectic manifold*, meaning that M is an even dimensional manifold equipped with a closed non-degenerate differential 2-form ω called the *symplectic form*. We suppose that $\pi_2(M) = 0$ and that ω is exact,

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BENOÎT JOLY, RWTH Aachen University, Pontdriesch 10–12, 52062 Aachen, Germany •
E-mail : `benoit.joly@imj-prg.fr`

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meaning that there exists a 1-form λ, called a *Liouville form*, which satisfies $d\lambda = \omega$.

Let us consider a time-dependent vector field $(X_t)_{t\in\mathbb{R}}$ defined by the equation

$$dH_t = \omega(X_t, \cdot), \tag{1}$$

where

$$\begin{aligned} H : \mathbb{R} \times M &\to \mathbb{R} \\ (t,x) &\mapsto H_t(x) \end{aligned}$$

is a smooth function 1-periodic on t, meaning that $H_{t+1} = H_t$ for every $t \in \mathbb{R}$. The function H is called a *Hamiltonian function.* If the vector field $(X_t)_{t\in\mathbb{R}}$ is complete, it induces a family $(f_t)_{t\in\mathbb{R}}$ of diffeomorphisms of M that preserve ω, also called *symplectomorphisms* or *symplectic diffeomorphisms*, satisfying the equation

$$f_0 = \mathrm{id} \text{ and } \frac{\partial}{\partial t} f_t(z) = X_t(f_t(z)).$$

In particular, the family $I = (f_t)_{t\in[0,1]}$ defines an isotopy from id to f_1. The map f_1 is called a *Hamiltonian diffeomorphism.* It is well known that the set of Hamiltonian diffeomorphisms of a symplectic manifold M is a group, which we denote $\mathrm{Ham}(M,\omega)$; we refer to [29] for more details.

Let us consider (M,ω) a symplectic manifold that is boundaryless, $\pi_2(M) = 0$, and such that ω is exact. We say that H is a *compactly supported Hamiltonian function* if there exists a compact set $K \subset M$ such that H_t vanishes outside K for every $t \in \mathbb{R}$. A compactly supported Hamiltonian function induces a *compactly supported Hamiltonian diffeomorphism* f. Such a map is equal to the identity outside a compact subset of M. Let us consider a compactly supported Hamiltonian diffeomorphism f and λ a Liouville form on M. The form $f^*\lambda - \lambda$ is closed because f is symplectic; but we have more, it is exact. More precisely, there exists a unique compactly supported function $A_f : M \to \mathbb{R}$, also called *action function*, such that

$$dA_f = f^*\lambda - \lambda.$$

In the literature, the *Calabi invariant* $\mathrm{Cal}(f)$ of f is defined as the mean of the function A_f, and we have

$$\mathrm{Cal}(f) = \int_M A_f \omega^n, \tag{2}$$

where $\omega^n = \omega \wedge \ldots \wedge \omega$ is the volume form induced by ω; see [29] for more details. We will prove later that the number $\mathrm{Cal}(f)$ does not depend on the choice of λ.

Let us give another equivalent definition of the Calabi invariant for a compactly supported Hamiltonian diffeomorphism f. We denote H a compactly supported Hamiltonian function defining f. The Calabi invariant of f can also be defined by the equation

$$\mathrm{Cal}(f) = (n+1)\int_0^1 \int_M H_t \omega^n dt. \tag{3}$$

To prove that $\int_M A_f \omega^n$ does not depend on the choice of the Liouville form λ, one may use the fact that the action function A_f satisfies

$$A_f(z) = \int_0^1 (\iota(X_s)\lambda + H_s)\circ f_s(z) ds, \tag{4}$$

where $(X_s)_{s\in\mathbb{R}}$ is the time dependent vector field induced by H by equation 1, and $(f_s)_{s\in\mathbb{R}}$ is the isotopy induced by the vector field $(X_s)_{s\in\mathbb{R}}$. Moreover, $\int_0^1 \int_M H_t \omega^n dt$ does not depend on the compactly supported Hamiltonian function H defining f.

The function Cal defines a real valued morphism on the group of compactly supported Hamiltonian diffeomorphisms of M, and thus it is a conjugacy invariant. It is an important tool in the study of difficult problems such as the description of the algebraic structure of the groups $\mathrm{Ham}(M,\omega)$. A. Banyaga proved in [4] that the kernel of the Calabi invariant is always simple, which means that it does not contain nontrivial normal subgroups. We also refer to [1] for more details about this definition.

In this article, we study the case of the dimension 2 and, more precisely, the case of the closed unit disk, which is a surface with boundary. We denote by $\|\cdot\|$ the usual Euclidian norm on $\mathbb{R}^2$, by $\mathbb{D}$ the closed unit disk and by $\mathbb{S}^1$ its boundary. The group of C^1 orientation preserving diffeomorphisms of $\mathbb{D}$ will be denoted by $\mathrm{Diff}^1_+(\mathbb{D})$. We consider $\mathrm{Diff}^1_\omega(\mathbb{D})$ the group of C^1 symplectomorphisms of $\mathbb{D}$, which preserve the normalized standard symplectic form $\omega = \frac{1}{\pi} du \wedge dv$, written in cartesian coordinates (u,v). In the case of the disk, the group $\mathrm{Diff}^1_\omega(\mathbb{D})$ is contractible, see [21] for a proof, and coincides with the group of Hamiltonian diffeomorphisms of $\mathbb{D}$. Moreover, the 2-form ω induces the Lebesgue probability measure denoted by Leb, and the symplectic diffeomorphisms are the C^1 diffeomorphisms of $\mathbb{D}$, which preserve the Lebesgue measure and the orientation.

Let us begin by the case of the unit open disk $\mathring{\mathbb{D}}$. The open disk is boundaryless and, hence, we already have two equivalent definitions of the Calabi invariant given by equations 2 and 3 on the set of compactly supported symplectic diffeomorphisms of $\mathring{\mathbb{D}}$. Let us give a third one. A. Fathi in his thesis [13] gave a dynamical definition, which is also described by J.-M. Gambaudo and É. Ghys in [17]: if we consider an isotopy $I = (f_t)_{t\in[0,1]}$ from id to f, there exists an angle function $\mathrm{Ang}_I : \mathring{\mathbb{D}} \times \mathring{\mathbb{D}} \setminus \Delta \to \mathbb{R}$ where Δ is the diagonal of $\mathring{\mathbb{D}} \times \mathring{\mathbb{D}}$ such that for each $(x,y) \in \mathring{\mathbb{D}} \times \mathring{\mathbb{D}} \setminus \Delta$, the quantity $2\pi \mathrm{Ang}_I(x,y)$ is the variation of angle of the vector $f_t(y) - f_t(x)$ between $t=0$ and $t=1$. If f is a